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Research Article

A Semi-Analytical Framework for higher-Order Delay Differential Equations: Utilizing Optimal Auxiliary Functions

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ABSTRACT

Delay differential equations (DDEs) are extensively utilized in fields such as control systems, biology, and engineering to model processes where current states depend on past states, effectively accounting for time lags. Key applications include population dynamics, epidemic modelling, and economic systems, where delayed responses significantly influence system behavior. This paper presents the extension of the Optimal Auxiliary Functions Method (OAFM) to second-order and third-order DDEs. The strength of this method lies in its convergence control parameters and auxiliary functions. Notably, the OAFM guarantees the convergence of approximate solutions after just one iteration, without requiring assumptions about small or large parameters. The method demonstrates both effectiveness and efficiency, with its accuracy validated through graphical and numerical results. Additionally, the results obtained are compared with those from the least squares method. Auxiliary functions and convergence control parameters are employed to further manage the convergence of the OAFM.

1. INTRODUCTION

Many relevant investigations in the areas of physics, engineering, biomathematics, and others are mathematically modelled using delay differential equations (DDEs). DDEs are differential equations in which the derivatives of certain unknown functions at two different time instants are correlated (the past and the present). Researchers in the engineering and bioscience fields commonly come across mathematical models based on DDEs [1].

In the modern era, Minorsky was the first researcher to explore the delay differential equations

$$\zeta'(\chi) = f\big(\chi, \zeta(\chi), \zeta(\chi - \tau)\big). \tag{1}$$

Typically, DDEs have been handled using discretization-based numerical techniques. This stems from these foundational techniques being suitably applied to solving first-order linear and simple non-linear DDEs. Due to their innately complex structure, DDEs are very difficult to study and if it is possible to achieve analytical solutions are achievable then they are implicit in nature [1].

The reason for studying DDEs is to introduce naturally appearing delays in the systems which gives models more life and realistic portrayal. This mathematically means DDEs have values which are dependent on previous solutions. Furthermore, time delays could be constant, dependent on time or state, or both [2].

The Delay Differential Equations (DDEs) were initially proposed in the 18th century by Laplace and Condorcet [3]. However, the theory and applications of those equations did not start to develop quickly until after World War II, and they have continued ever since. In 1942, Pontryagin developed the fundamental theory governing the stability of systems defined by these kinds of equations. Smith published an important book in 1957, Pinny in 1958, Bellman and Cooke in 1963, Halanay in 1966, Myshkis in 1972, Hale in 1977, Yanusherski in 1978, and Marshal in 1979 all wrote significant publications [4].

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To further motivate the study of DDEs, there are many physical and technical systems which include intrinsic delays, also known as heredity or memories, retarded arguments after actions, dead times, or time lags [5]. It may be challenging to include time delays in mathematical models because pure delays are frequently employed to illustrate the impact of transmission, transportation, and initial phenomena [4]. Therefore, they provide an effective model for a wide range of phenomena in the applied sciences, including population dynamics, infectious disease, physiological and pharmaceutical kinetics, chemical kinetics, models of conveyor belts, urban traffic management, heat exchangers, robotics, navigational control of ships and aircraft, control theory, mathematical biology, mathematical economics, biochemical, medical, control system, biological models, and more general phenomena [6-9]. Delay equations used by Bunsenberg et al. [10] for modeling the embryonic cell cycle. Patel et al. [11] proposed an iterative scheme for the optimal control systems with a quadratic cost functional. By using DDEs, the input-to-state stability of a time-invariant system with numerous non-commensurate and dispersed time delays as well as HIV-1 therapy for combating one virus with another, both have been recently modeled [7,12]. DDEs become particularly important when the model based on ordinary differential equations fails.

There are a few classes of nonlinear ODEs for which solutions are easy to find despite the obvious connections between ODE and DDEs. In contrast, there may be a number of distinct and significant ways in which the solutions to DDEs problems and those to ODE problems diverge [14]. There is always a vast range of frequencies produced by delay problems. Numerical techniques, asymptotic solutions, approximations, and graphical approaches are employed to solve them. Due to a significant increase in the use of delay models, several authors have examined and proposed numerous methods for solving DDEs, Variational Iteration Method [14], Spline methods [15], Optimal Auxiliary Function Method [16-17], Homotopy Analysis Method [18], Homotopy Perturbation Method (HPM) [19], Adomain Decomposition Method [20], Kudryashov's method [21], Modified Variational Iteration Algorithm-II [22], Iterative Decomposition Method [23], the Runge-Kutta Method [24], Reproducing Kernel Method [25], the Hermite Interpolation Method [26], the Variable Multistage Method [27], the Direct Block One Step Method [28], B-spline Collection Method [29], the Direct Two And Three-Point One-Step Block Method [30] etc.

The numerical solution of DDEs is extremely intriguing, and many techniques have been used to solve particular equations. To demonstrate the analytical solution of homogeneous DDEs, Asl et al. [31] used Lambert functions. Also, Bellen et al. [32] described techniques for Gaussian points based on the predictor-corrector version of the one-step collocation method for non-stiff DDEs with time-dependent delays. Ismail et al. [33] compared the numerical results based on Newton Divided Difference and In't Hout interpolations, in order to solve delay differential equations. Martin et al. [34] introduced variable step size multistep methods. Evans et al. [35] used the Adomian decomposition method and proposed a numerical method for linear and non-linear Higher Order Delay Differential Equations. While, Taiwo et al. [36] used an elementary decomposition method for solving Delay Differential Equations. In the scenario involving Constant and Variable Coefficients, Olvera et al. [37] expanded the enhanced the Multistage Homotopy Perturbation Method (EMHPM).

A series of authors have used transformation methods to solve the problem, the Differential Transform Method by Liu et al. for Delay Differential Equations [38], with Shampine et al. [39] suggested a numerical solution as well. Aboodh et al. [40] used the Aboodh Transformation method, Ebimene et al. [41] applied Elzaki Transformation Technique, and Yaman et al. [42] used Daftardar-Jafari Method for solving Nonlinear Delay Differential Equations. Finally, the Sumudu Transform method (STM) was used to solve a generic form of delay differential equations of the pantograph type [43].

In the literature, there have been many alternative techniques used as well for BVPs such as HPM by Bellen and Aslamnoor et al. [44] for DDE BVPs. The Laplace Adomian Decomposition Method (LADM) [45] for a second order of DDE BVP by Kanth et al. More recently, Anakira et al. [46] expanded the applicability of the Optimal Homotopy Asymptotic Method (OHAM), in order to obtain the approximate analytic solution of DDEs. The dynamics of cutting machine operations were modeled by the stability lobes of DDEs, on the other hand, were determined by Insperger et al. [47] using the semi-discretization method. Based on the characteristics of the Chebyshev polynomials, Butcher et al. [48] created a method to obtain the stability lobes of milling machine operations and they demonstrated that this method is quicker than the full and semidiscretization methods because these solution techniques are approximations to the original DDEs by a series of ODEs [49]. To solve Delay Differential Equations, Adomian Decomposition Method (ADM) was utilized by Blanco et al. [50], Homotopy Perturbation Method (HPM) was used by Biazer et al. [51], the Homotopy Analysis Method (HAM) were studied by Alomari et al. [52]. In addition to particular types of equations, Predictor-corrector methods were examined by Bhalekar et al. [53]. For pantograph DDEs, the Residual Power Series Method was used by the authors of [54]. In [56- 58], there are many more numerical methods such as Galerkin and DDEs applications are discussed.

Stability, existence, and uniqueness of DDEs have been initially addressed in the works of [59-61]. The existence and uniqueness of DDEs were examined by Eloe et al. [62] and Rebenda et al. [63] solves DDEs by the extension of semi-analytical technique. The method of DTM was used by Mohammed et al. [64], Mirzaee et al. [65] and Rostam et al. [66] to get at a numerical solution to DDEs. Verleydeu presented the collection method with an iterative linear system solver in 2003 [67] in order to compute the solution of a system of autonomous Delay Differential Equations. By using the direct Ritz method, Ordinary Delay Differential Equations were generalized to Partial Delay Differential Equations in 2004 and solved the variational formulation of the specific forms of partial delay differential equations [68]. In 2007, Luo [69] studied the exponential stability of nth order Delay Differential Equations. In 2006, Forde, in his doctoral dissertation [70] examined the modelling and stability of some biological systems as Delay Differential Equations. In 2001, Caus V. et al. [71] using non-polynomial spline functions investigated the numerical stability and convergence of the numerical solution of Delay Differential Equations. Using the method of steps and the Laplace Transformation Method in 2009, Nagy T.K. studied the solution and stability of Delay Differential Equations [72].

In summary, delay differential equations (DDEs) play a crucial role in modeling real-world systems where current states are influenced by past states, capturing the dynamics of processes across various fields such as engineering, biology, and economics. Despite their significance, there remains a gap in the application of efficient and robust analytical methods for solving second-order and third-order DDEs. The Optimal Auxiliary Functions Method (OAFM) is introduced to address this gap, providing a systematic approach to obtain solutions that can enhance understanding and predictability in systems characterized by time delays.

Novelty

- **First-Time Application**: This paper presents the first application of the Optimal Auxiliary Functions Method (OAFM) to second-order and third-order Delay Differential Equations, marking a significant advancement in the methodology available for analyzing these equations. To the best of the author's knowledge, no one has yet applied the Optimal Auxiliary Functions Method (OAFM) to solve delay differential equations in the literature.
- **Expanded Research Base**: The method's successful application to a range of mathematical problems, such as Partial Differential Equations and the SEIR epidemic model, highlights its versatility and effectiveness in addressing both linear and nonlinear dynamics.

In Section 2, the Basic Idea of OAFM is briefly reviewed, and in Section 3, the governing equations are examined by addressing the approximations to the solutions of the second and third order delay differential equations. Section 4 also includes results and discussions. Finally, Section 5 discusses the conclusion.

2. BASIC IDEA OF OAFM

The general form of nonlinear differential equation is given below:

$$L(\zeta(\chi)) + N(\zeta(\chi)) + G(\chi) = 0, \qquad (2)$$

in which an unknown function is $\zeta(\chi)$, a linear operator is L, a nonlinear operator N, and a source operator G. The initial or boundary conditions are given below

$$B\left(\zeta(\chi), \frac{d\zeta(\chi)}{d\chi}\right) = 0. \qquad (3)$$

We require an approximate solution $\tilde{\zeta}(\chi)$ for Eq. (2) and Eq. (3), with only two components:

$$\tilde{\zeta}(\chi) = \zeta_0(\chi) + \zeta_1(\chi, C_i),\ i = 1,2,\dots,n \qquad (4)$$

Where C_i from $i = 1,2,\dots,n$ are currently unknown parameters. Putting Eq. (4) into Eq. (2), we get

$$L[\zeta_0(\chi)] + L[\zeta_1(\chi, C_i)] + N[\zeta_0(\chi) + \zeta_1(\chi, C_i)] + G(\chi) = 0. \qquad (5)$$

To find the initial approximation $\zeta_0(\chi)$, the linear equation can be used

$$L[\zeta_0(\chi)] + G(\chi) = 0, \qquad B\left(\zeta_0(\chi), \frac{d\zeta_0(\chi)}{d\chi}\right) = 0 \qquad (6)$$

The first approximation, yields from Eq. (6), the following equation

$$L[\zeta_1(\chi, C_i)] + N[\zeta_0(\chi) + \zeta_1(\chi, C_i)] = 0, \quad B\left(\zeta_1(\chi_i), \frac{d\zeta_1(\chi)}{d\chi}\right) = 0. \qquad (7)$$

In general, Eq. (7) is a nonlinear differential equation that is hard to solve. The nonlinear term from Eq. (7) is build up into the form at this point. Notice that the second term can be estimated as

$$N[\zeta_0(\chi) + \zeta_1(\chi, C_i)] = N[\zeta_0(\chi)] + \sum_{k=1}^{n} \frac{\zeta_1^k(\chi, C_i)}{k!} N^{(k)}[\xi_0(\chi)], \qquad (8)$$

where $n \rightarrow \infty$\ and $N^{(k)} = \frac{d^k N}{d\chi^k}$. To resolve the difficulties \ arise in solving the nonlinear differential equation (7) from using equation (8) to accumulate the convergence of the first approximate solution $\tilde{\zeta}(\chi, C_i)$, we represent eq. (7) with an alternative form of eq. (8)

$$N[\zeta_0(\chi) + \zeta_1(\chi, C_i)] = N[\zeta_0(\chi)] + A_1(\zeta_0(\chi, C_j))F[N(\zeta_0(\chi))] + A_2(\zeta_0(\chi, C_k))$$

Resulting in

$$L[\zeta_1(\chi, C_i)] + A_1(\zeta_0(\chi), C_j)F[N(\zeta_0(\chi))] + A_2(\zeta_0(\chi), C_k) = 0,\ B\left(\zeta_1(\chi), \frac{d\zeta_1(\chi)}{d\chi}\right) = 0 \tag{9}$$

Where A_1 and A_2 are resultory auxiliary functions, and are selected based on the initial approximation $\zeta_0(\chi)$, or $N[\zeta_0(\chi)]$, or in a combination of $\zeta_0(\chi)$ and $N[\zeta_0(\chi)]$. The C_j and C_k respectively are various unknown parameters with $j = 1,2,..,p$, $k = p+1, p+2, \ldots, n, i = j + k$. and $F[N(\zeta_0(\chi))]$ is the operator component of $N[\zeta_0(\chi)]$.

Minimizing the square residual error is one of the techniques that can be used to determine the unknown parameters C_j and $C_k$as accurately as possible

$$J\ (C_i, C_k) = \int_{(D)} R^2\ (\chi, C_i, C_k) d\chi, \tag{10}$$

Where $R\ (\chi, C_i, C_k) = L[\tilde{\zeta}_1(\chi, C_i)] + N[\tilde{\zeta}(\chi, C_i)] + G(\chi), i = j + k, j = 1,2, \ldots, p, k = p + 1, \ldots, n.$

The residual minimization is subject to the following conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_n} = 0. \tag{11}$$

The convergence-control parameters can also be obtained by using the Ritz method, Galerkin method, Kantowich method, and collocation methods, etc.

Remark: The auxiliary functions can exist in either form $\zeta_0(\chi)$ or form $N[\zeta_0(\chi)]$, or they can combine both forms.

- If $\zeta_0(\chi)$ or $N[\zeta_0(\chi)]$ a polynomial function then the auxiliary functions should be the sum of polynomial functions.
- The auxiliary functions should be the sum of polynomial functions if $\zeta_0(\chi)$ or $N[\zeta_0(\chi)]$ is a polynomial function.
- If $\zeta_0(\chi)$ or $N[\zeta_0(\chi)]$ are trigonometric, the auxiliary functions should equal the sum of the trigonometric functions.

3. GOVERNING EQUATIONS

To demonstrate the effectiveness and precision of the suggested method, we offer approximate solutions for second order and third order delay differential equations in this section. Mathematica 10 is used to perform all computations.

Example 1. Take the second order delay differential equations, for instance [36]

$$\frac{d^2\zeta}{d\chi^2} = \frac{3}{4}\zeta(\chi) + \zeta\left(\frac{\chi}{2}\right) - \chi^2 + 2, 0 \leq \chi \leq 1, \tag{12}$$

where the given initial condition is

$$\zeta(0) = 0, \zeta'(0) = 0. \tag{13}$$

The exact solution to equation (12) is given in [36], which is

$$\zeta(\chi) = \chi^2. \tag{14}$$

From eq. (12), linear and nonlinear expressions are given

$$\begin{cases} L(\zeta(\chi)) = \frac{d^2\zeta}{d\chi^2}, \\ N(\zeta(\chi)) = -\frac{3}{4}\zeta(\chi) + \zeta\left(\frac{\chi}{2}\right), \\ G(\chi) = \chi^2 - 2. \end{cases} \tag{15}$$

We obtain the following zero order problem by applying the OAFM described in section (2):

$$\frac{d^2\zeta_0}{d\chi^2} - \chi^2 + 2 = 0, \zeta_0(0) = 0, \zeta_0'(0) = 0. \tag{16}$$

The solution for eq. (16) is given,

$$\zeta_0(\chi) = \frac{1}{12}(12\chi^2 + \chi^4). \tag{17}$$

If eq. (17) is substituted for the nonlinear part of eq. (15), we obtain

$$N(\zeta_0) = -\chi^2 - \frac{13\chi^4}{192}. \tag{18}$$

By using OAFM, the first order problem is,

$$\frac{d^2\zeta_1}{d\chi^2} - A_1 N(\zeta_0) + A_2 = 0, \zeta_0(0) = 0, \zeta_0'(0) = 0, \tag{19}$$

We select the auxiliary functions A_1, A_2 in the following way,

$$A_1 = C_1\left(-\chi^2 - \frac{13\chi^4}{192}\right) + C_2\left(-\chi^2 - \frac{13\chi^4}{192}\right)^2 + C_3\left(-\chi^2 - \frac{13\chi^4}{192}\right)^4 + C_4\left(-\chi^2 - \frac{13\chi^4}{192}\right)^6,$$
$$A_2 = 0. \tag{20}$$

The solution to eq. (19) is obtained by substituting eq. (18) and (20) into eq (19);

$$\zeta_1(\chi, C_i) = \begin{pmatrix} \frac{C_1\chi^6}{30} + \frac{(-13C_1+96C_2)\chi^8}{5376} - \frac{13(13C_1-576C_2)\chi^{10}}{3317760} + \\ \frac{(169C_2+12288C_3)\chi^{12}}{1622016} + \frac{(169C_2+184320C_3)\chi^{14}}{99090432} + \\ \frac{(845C_3+18432C_4)\chi^{16}}{4423680} + \frac{13(845C_3+129024C_4)\chi^{18}}{1082916864} + \\ \frac{169(845C_3+774144C_4)\chi^{20}}{516402708480} + \frac{2197(169C_3+1290240C_4)\chi^{22}}{120544699613184} \\ + \frac{999635C_4\chi^{24}}{750142881792} + \frac{199927C_4\chi^{26}}{4348654387200} + \frac{4826809C_4\chi^{28}}{5410421842378752} + \\ \frac{62748517C_4\chi^{30}}{8368119116212469760} \end{pmatrix}. \tag{21}$$

By combining eq. (17) and eq. (19), the first order approximate solution by OAFM can be found (21),

$$\tilde{\zeta}(\chi) = \zeta_0(\chi) + \zeta_1(\chi, C_1, C_2, C_3, C_4) \tag{22}$$

$$\tilde{\zeta}(\chi, C_i) = \begin{pmatrix} \frac{1}{12}(12\chi^2 + \chi^4) + \frac{C_1\chi^6}{30} + \frac{(-13C_1+96C_2)\chi^8}{5376} - \frac{13(13C_1-576C_2)\chi^{10}}{3317760} + \\ \frac{(169C_2+12288C_3)\chi^{12}}{1622016} + \frac{(169C_2+184320C_3)\chi^{14}}{99090432} + \frac{(845C_3+18432C_4)\chi^{16}}{4423680} \\ + \frac{13(845C_3+129024C_4)\chi^{18}}{1082916864} + \frac{169(845C_3+774144C_4)\chi^{20}}{516402708480} + \frac{999635C_4\chi^{24}}{750142881792} \\ + \frac{2197(169C_3+1290240C_4)\chi^{22}}{120544699613184} \frac{199927C_4\chi^{26}}{4348654387200} + \frac{4826809C_4\chi^{28}}{5410421842378752} + \\ \frac{62748517C_4\chi^{30}}{8368119116212469760} \end{pmatrix}. \tag{23}$$

Therefore, to calculate the exact values of the convergence control parameters $C_i, i = 1,2,3\ldots$, use the least squares method, as stated in eq. (23)

$$C_1 = 7.43380999587875, \quad C_2 = 12.046912067424115,$$
$$C_3 = -9.2979628507085, \quad C_4 = 3.5379687045401496. \tag{24}$$

The first order OAFM solution is given as, after substitution of eq. (24) in eq. (23),

$$\tilde{\zeta}(\chi, C_i) = \begin{pmatrix} \chi^2(1. + 0.0833333333333333\chi^2 - 0.24779366652929166\chi^4 + 0.19714732 \\ 67348012\chi^6 + 0.026810547981640703\chi^8 - 0.06918392874676438\chi^{10} - 0 \\ .017274771639942003\chi^{12} + 0.01296546326887011\chi^{14} + 0.005385583543 \\ 590115\chi^{16} + 0.0008937711425761287\chi^{18} + 0.00008316812359553707\chi^{20} \\ +0.000004714671606980128\chi^{22} + 1.626561704408144 \times 10^{-7}\chi^{24} + 3.1563 \\ 34142197792 \times 10^{-9}\chi^{26} + 2.65295326607142 \times 10^{-11}\chi^{28}) \end{pmatrix}.$$

Table 1: Absolute error comparison for Problem 1 between the OAFM and the decomposition method.

χ	OAFM Solution	Exact Solution.	Abs Errors [36]	Abs Error OAFM
0.	0.	0.	0.	0.
0.1	0.0100	0.01	2.6909×10^{-5}	8.0875×10^{-6}
0.2	0.0401	0.04	1.0763×10^{-4}	1.1798×10^{-4}
0.3	0.0905	0.09	2.4218×10^{-4}	5.0741×10^{-4}
0.4	0.1612	0.16	4.3056×10^{-4}	1.2491×10^{-3}
0.5	0.2521	0.25	6.7277×10^{-4}	2.1151×10^{-3}
0.6	0.3625	0.36	9.6890×10^{-4}	2.5524×10^{-3}
0.7	0.4919	0.49	1.3191×10^{-3}	1.9559×10^{-3}
0.8	0.6400	0.64	1.7237×10^{-3}	8.9172×10^{-5}
0.9	0.8070	0.81	2.1837×10^{-3}	2.9623×10^{-3}
1	0.9923	1.	2.7003×10^{-3}	7.6282×10^{-3}

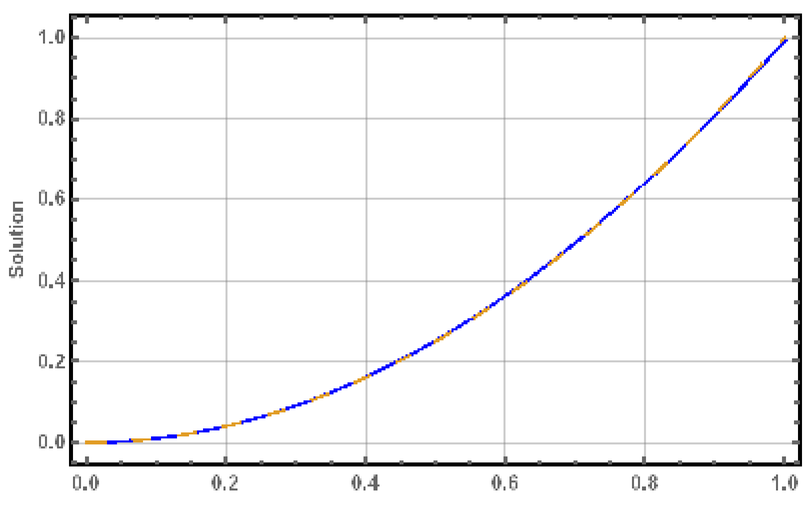

Figure 1. The 2D graph produced by the exact verses OAFM solution to Problem 1.

Example 2

Consider the third order delay differential equation [36]

$$\frac{d^3\zeta}{d\chi^3} = -1 + 2\zeta^2\left(\frac{\chi}{2}\right), 0 \le \chi \le 1, \tag{25}$$

subject to the initial condition

$$\zeta_0(0) = 0, \zeta_0'(0) = 1, \zeta_0''(0) = 0. \tag{26}$$

The exact solution to equation (25) is given in [36], which is

$$\xi(\chi) = sin(\chi). \tag{27}$$

From eq. (25), the linear and nonlinear expressions are

$$\begin{cases} L(\zeta(\chi)) = \frac{d^3\zeta}{d\chi^3}, \\ N(\zeta(\chi)) = -2\zeta^2\left(\frac{\chi}{2}\right), \\ G(\chi) = 1. \end{cases} \tag{28}$$

Using the OAFM stated in section (2), we arrive at the following zero-order problem:

$$\zeta_0(\chi) = \frac{1}{6}(6\chi - \chi^3). \tag{29}$$

Here, we choose A_1, A_2based on the first operator's non-linear operator.

$$A_1 = C_1\left(\frac{\chi^4}{8}\right) + C_2\left(\frac{\chi^4}{8}\right)^2,$$
$$A_2 = C_3\left(\frac{\chi^4}{8}\right)^4 + C_4\left(\frac{\chi^4}{8}\right)^6. \tag{30}$$

We obtain the first-order solution by applying the OAFM method outlined for Problem 1:

$$\zeta_1(\chi) = \frac{1}{114422199091200}\begin{pmatrix} 14189260800C_1\chi^9 - 601968640C_1\chi^{11} + 7235200C_1\chi^{13} \\ +520934400C_2\chi^{13} - 27287040C_2\chi^{15} + 380380C_2\chi^{17} - \\ 4804800C_3\chi^{19} - 24871C_4\chi^{27} \end{pmatrix}. \tag{31}$$

We combine Eqs. (30) and (32) to get the OAFM solution of the first order:

$$\tilde{\zeta}(\chi, C_i) = \begin{pmatrix} \chi - \frac{\chi^3}{6} + \frac{C_1\chi^9}{8064} - \frac{C_1\chi^{11}}{190080} + \frac{(C_1 + 72\,C2)\,\chi^{13}}{15814656} - \frac{C_2\chi^{15}}{4193280} \\ + \frac{C_2\chi^{17}}{300810240} - \frac{C_3\chi^{19}}{23814144} - \frac{C_4\chi^{27}}{4600627200} \end{pmatrix}. \tag{32}$$

The least squares method is used to determine the convergence control parameters found in equation (32). Eq. (33) provides the numerical values. These numbers in eq. (32) give us the first-order approximation of the answer to problem 2, shown in fig. 2.

$$C_1 = -64437.9111676810, C_2 = 3339289.387826368,$$
$$C_3 = 2.433012058446148, C_4 = -1.30654767236302. \tag{33}$$

Table 2: Third order delay differential equations for Problem 2, approximate solution found by OAFM.

χ	OAFM Solution	Exact Sol [36]	Abs Error OAFM
0.	0.	0.	0.
0.1	0.0998	0.0998	9.1299×10^{-8}
0.2	0.1986	0.1986	6.7360×10^{-6}
0.3	0.2951	0.2955	1.7447×10^{-4}
0.4	0.3821	0.3894	2.0646×10^{-3}
0.5	0.4655	0.4794	1.3888×10^{-2}
0.6	0.43216	0.56464	0.13247×10^{-1}

Table 3: Absolute error for the different numbers of the convergence control parameter obtained by OAFM.

χ	OAFM Solution	Exact Solution	Absolute Error using C_1 and C_2	Absolute Error using C_1, C_2 and C_3	Absolute Error using C_1, C_2, C_3 and C_4
0.	0.	0.	0.	0.	0.
0.1	0.09982	0.09983	6.02843×10^{-6}	1.35998×10^{-7}	9.12993×10^{-8}
0.2	0.19832	0.19866	3.42256×10^{-4}	1.60182×10^{-5}	6.73605×10^{-6}
0.3	0.29209	0.29552	3.42351×10^{-3}	3.53646×10^{-4}	1.74478×10^{-4}
0.4	0.37279	0.38941	1.66264×10^{-2}	3.24894×10^{-3}	2.06466×10^{-3}
0.5	0.42569	0.47942	5.37309×10^{-2}	1.75191×10^{-2}	1.38887×10^{-2}
0.6	0.43216	0.56464	0.13247×10^{-1}	6.49447×10^{-2}	6.10830×10^{-2}

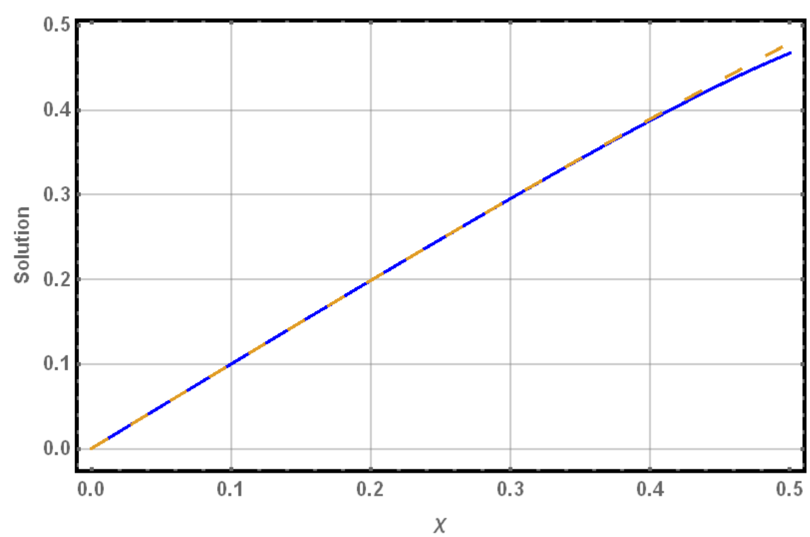

Figure 2. The 2D graph produced by the exact verses OAFM solution to Problem 2.

Example 3

Consider the second order Nonlinear proportional delay differential equation [42]

$$\frac{d^2\zeta}{d\chi^2} = -\zeta(\chi) + 5\zeta^2\left(\frac{\chi}{2}\right), \quad 0 \le \chi \le 1 \tag{34}$$

subject to the initial condition given by,

$$\zeta_0(0) = 1, \zeta_0^{'}(0) = -2. \tag{35}$$

The exact solution to eq. (34) is given in [42], which is

$$\zeta(\chi) = e^{-2\chi}. \tag{36}$$

The auxiliary functions A_1, A_2 can be choose for Example 3:

$$A_1 = C_1(cos(\chi)) + C_2(cos(\chi))^2 + C_3(cos(\chi))^4 + C_4(cos(\chi))^6,$$
$$A_2 = 0. \tag{37}$$

then using the same procedure as discussed in Example 2, we get zero-order and the first order OAFM solution for Example 3:

$$\zeta_0(\chi) = cos(\chi) - 2\,sin(\chi). \tag{38}$$

$$\zeta_1(\chi, C_i) = \begin{pmatrix} 161.976 + (10.4752 - 51.2296\,\chi)\chi + 24.0056\cos(2\chi) - 1.71599\cos(3\chi) - \\ 0.3777\cos(4\chi) + 0.0677949\cos(5\chi) + 0.0109339\cos(6\chi) - 0.00240993 \\ \cos(7\chi) - 28.1586\sin(\chi) + \cos(\chi)\,(-183.964 + 25.6664\sin(\chi)) - 2.8251 \\ 5\sin(3\chi) + 0.102989\sin(5\chi) - 0.00321323\sin(7\chi) \end{pmatrix}. \tag{39}$$

By using OAFM, the first order problem is obtained by combining eq. (38) and eq. (39),

$$\tilde{\zeta}(\chi, C_i) = \begin{pmatrix} (1/1128960)(-4821600\,C_2 - 3732918\,C_3 - 3147569\,C_4 + 840 \\ (\chi\,(-128\,(35\,C_2 + 21\,C_3 + 15\,C_4) + 525(8\,C_2 + 6\,C_3 + 5\,C_4)\,\chi) - \\ 420\,C_1(-37 + 8\chi + 6\chi^2)) - 8820\,(1600\,C_1 - 8\,\,(16 + 90\,C_2 + 75\,C_3 \\ ) - 525\,C_4)\,\,\cos(\chi) + 22050\,(48\,C_1 - 80\,\,(C_2 + C_3) - 75\,C_4)\cos(2 \\ \chi) + 14700\,(16\,C_2 + 20\,C_3 + 21\,C_4)\cos(3\chi) - 55125\,(2\,C_3 + 3 \\ C_4)\cos(4\chi) + 5292\,(4\,C_3 + 7\,C_4)\cos(5\chi) - 12250\,C_4\cos(6\chi) + \\ 2700\,C_4\cos(7\chi) - 2257920\sin(\chi) + 176400\,(16\,C_2 + 8\,C_3 + 5 \\ C_4)\sin(\chi) + 1411200\,C_1\sin(2\chi) + 19600(16\,C_2 + 12\,C_3 + 9\,C_4 \\ )\sin(3\chi) + 7056(4\,C_3 + 5\,C_4)\sin(5\chi) + 3600\,C_4\sin(7\chi)) \end{pmatrix}. \tag{40}$$

The least squares method is used to determine the convergence control parameters found in equation (40). Eq. (41) provides the numerical values. These numbers in eq. (40) give us the first-order approximation of the answer to problem 2, shown in fig. 2.

$$C_1 = 10.26654949107283, \quad C_2 = -13.63809934533353,$$
$$C_3 = 5.379148619033602, \quad C_4 = -1.007669976790007. \tag{41}$$

Table 4: Approximate solution, for nonlinear second order proportional Delay differential equations for Problem 3, obtained by OAFM.

χ	OAFM Solution	Exact solution [42]	Abs Error OAFM
0	1.	1.	1.1990×10^{-14}
0.1	0.8187	0.8187	1.2464×10^{-6}
0.2	0.6703	0.6703	2.9469×10^{-6}
0.3	0.5488	0.5488	4.6849×10^{-6}
0.4	0.4493	0.4493	6.4888×10^{-6}
0.5	0.3678	0.3678	8.9538×10^{-6}
0.6	0.3011	0.3011	1.4104×10^{-5}
0.7	0.2466	0.2465	1.7193×10^{-5}
0.8	0.2022	0.2018	3.3110×10^{-4}
0.9	0.1670	0.1652	1.7483×10^{-3}
1.	0.1415	0.1353	6.2440×10^{-3}

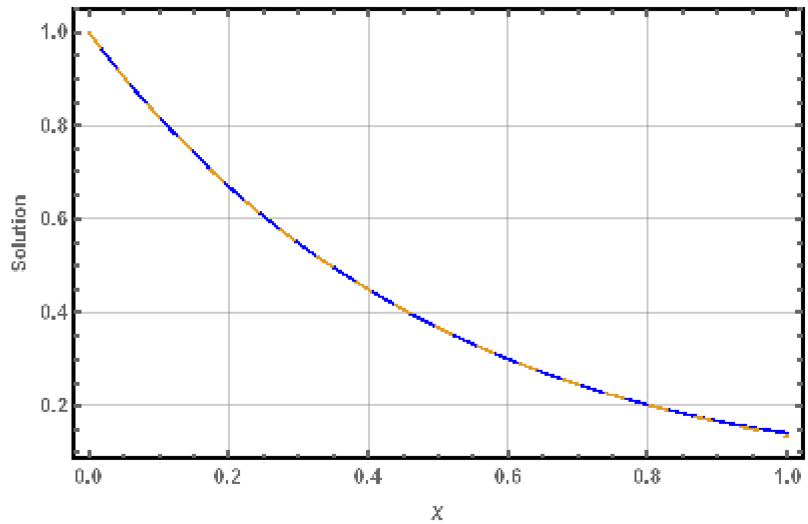

Figure 3. The 2D graph produced by the exact verses OAFM solution to Problem 3.

4. RESULTS AND DISCUSSIONS

Without the use of small parameter assumptions or discretization, the numerical problems of the formulation shown in section 3 and the extension of the (OAFM) scheme for (DDEs) presented in section 2 provide a highly accurate solution for the difficulties at hand, find the numerical approximation solutions to the second order and third order delay differential equations for problems 1-3. Figure 1 shows the approximate solution by OAFM for the exact solution $\zeta(\chi)$ and the second order approximate solution $\tilde{\zeta}(\chi, C_i)$. Comparisons of the absolute errors between the suggested method and the exact solution are shown in Table 1. The OAFM is seen to converge to exact solution. Figure 2 shows the approximate solution via OAFM for the exact solution $\zeta(\chi)$ and the third order approximate solution $\tilde{\zeta}(\chi, C_i)$. The approximate and exact solutions are shown in Table 2 respectively. Table 3 shows the Absolute error for the different numbers of the convergence control parameter obtained by OAFM, which shows that increasing the number of convergence control parameters, the OAFM converged rapidly to exact solution. The numerical values of the convergence control parameters are calculated using the least square method. Figure 3 shows the approximate solution via OAFM for the exact solution $\zeta(\chi)$ and the second order approximate solution $\tilde{\zeta}(\chi, C_i)$. The approximate and exact solutions are shown in Table 4 respectively. The OAFM is seen to converge to exact solution. The numerical values of the convergence control parameters are calculated using the collocation method.

5. CONCLUSION

The Optimal Auxiliary Functions Method (OAFM) has been applied for the first time to solve second-order and third-order delay differential equations without requiring assumptions about small or large parameters. This innovative approach provides a more generalizable solution framework, yielding highly accurate numerical approximations when compared to exact solutions. The OAFM demonstrates efficient convergence, achieving rapid convergence to the exact solution after just one iteration, which highlights its effectiveness. The method utilizes auxiliary functions and carefully selected convergence control parameters to ensure reliable convergence. Results are illustrated through figures and tables, which show the effectiveness of the OAFM by comparing approximate solutions with exact solutions, revealing low absolute errors. Moreover, the OAFM effectively addresses strong nonlinear problems, showcasing its robustness compared to traditional methods. Notably, the OAFM offers significant advantages over other perturbation and numerical methods by eliminating the need for discretization, simplifying the implementation process. Increasing the number of convergence control parameters further enhances the accuracy and speed of convergence to the exact solution. The numerical values of these convergence control parameters are calculated using both collocation and least squares methods, providing a comprehensive analysis of the approach.

Conflicts of Interest

The authors declare no conflicts of interest.

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