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Review Article

Review Note on Hermite-Hadamard Type Integral Inequality Via Riemann-Liouville Fractional Integral Operators

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ABSTRACT

In the context of fractional calculus, the concept of convexity is primarily used to tackle various challenges in both theoretical and applied research. This review paper aims to present Hermite–Hadamard (H-H) inequalities related to different classes of convex functions via Riemann-Liouville fractional integral operators.

1. INTRODUCTION

Convex functions have a long and illustrious history. The history of convexity theory can be traced all the way back to the end of the nineteenth century. Convex theory provides us with appropriate guidelines and techniques to focus on a broad range of problems in applied sciences. The establishment of multiple sections in contemporary mathematics, such as engineering, financial mathematics, economics, and optimization, has been significantly aided by convexity theory. In recent years, mathematical inequalities have been widely acknowledged to have contributed to the development of various aspects of mathematics, as well as other scientific disciplines. The construction of mathematical models is the main intention of fractional calculus. When demonstrating the uniqueness of solutions for specific fractional partial differential equations, fractional integral inequalities are crucial. Additionally, they offer upper and lower bounds for solutions of fractional boundary value problems. The improvement of popular integral inequalities in the frame of fractional integral operators has been a stimulating field of mathematical research in recent years. The goal of this study is to integrate the notion of integral inequalities into the framework of Riemann-Liouville fractional integral operators, which are of particular interest because of their characteristics and frequent use. Outside of mathematics, the issue has received a lot of interest from subjects like game theory, variational methods, mathematical economics, mathematical programming, operation research, probability, and statistics.

This review summarizes the existing knowledge on convexity and enhances the reader's understanding by discussing findings from recent research. Our objective is to provide a comprehensive and thorough review, including as many results as possible to showcase progress in the field. While lengthy proofs are omitted for brevity, readers are directed to the relevant articles for these details.

List of Symbols

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The following symbols are used in this research proposal:
$\Gamma(\alpha)$ Gamma function
Γ_k k-Gamma function.
$B(\mathscr{d}_1, \mathscr{d}_2)$ Beta function

Abbreviations
The following abbreviations are used in this research proposal:
CF Convex function
H-CF Harmonic Convex function
H–H Hermite–Hadamard
R-LFIO Riemann–Liouville fractional integral operator
k-R-LFIO k-Riemann–Liouville fractional integral operator

1.1 Convex Function

Definition 1. *[1]*
Let $\Theta : \mathbb{I} \to \mathbb{R}$ *be a function, then an inequality of the form*

$$\Theta\left(\wp \mathscr{d}_1 + (1-\wp)\mathscr{d}_2\right) \leq \wp\Theta\left(\mathscr{d}_1\right) + (1-\wp)\Theta\left(\mathscr{d}_2\right), \tag{1}$$

is said to be convex, if $\forall\ \mathscr{d}_1, \mathscr{d}_2 \in \mathbb{I}$ *and* $\wp \in [0, 1]$.

1.2 Hermite-Hadamard Inequality

The concept of inequality is one of the key concept in the field of applied sciences, which has an abundance of intriguing applications. Several improvements on the famous Hermite-Hadamard inequalities have been addressed and developed for modifications on convex functions in this way. This inequality was introduced & investigated by Hermite [2]and Hadamard [3] respectively in 1893. This inequality is an identification of convex functions that has been widely explored. This inequality assert that:
If $\Theta : \mathbb{I} \to \mathbb{R}$ is CF in $\mathbb{I}$ for $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{I}$ and $\mathscr{d}_1 < \mathscr{d}_2$, then

$$\Theta\Big(\frac{\mathscr{d}_1 + \mathscr{d}_2}{2}\Big) \leq \frac{1}{\mathscr{d}_2 - \mathscr{d}_1} \int_{\mathscr{d}_1}^{\mathscr{d}_2} \Theta(\wp) d\wp \leq \frac{\Theta(\mathscr{d}_1) + \Theta(\mathscr{d}_2)}{2}. \tag{2}$$

2. H-H TYPE INEQUALITY VIA RIEMANN-LIOUVILLE FRACTIONAL INTGERAL OPERATOR

Definition 2 ([4])**.** *Assume that* $\Theta \in L_1[\mathtt{u}_1, \mathtt{u}_2]$. *Then, the R-LFIO* $J^{\alpha}_{\mathtt{u}_1+}\Theta$ *and* $I^{\alpha}_{\mathtt{u}_2-}\Theta$, $\alpha > 0, \mathtt{u}_1 \geq 0$ *are defined by*

$$I^{\alpha}_{\mathtt{u}_1+}\Theta(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathtt{u}_1}^{x} (x - \wp)^{\alpha-1} \Theta(\wp) d\wp, \quad x > \mathtt{u}_1,$$

and

$$I^{\alpha}_{\mathtt{u}_2-}\Theta(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mathtt{u}_2} (\wp - x)^{\alpha-1} \Theta(\wp) d\wp, \quad x < \mathtt{u}_2,$$

respectively.

In 2013, Sarikaya et al. [5] investigated a novel generalization of H-H inequality via R-LFIO involving CF.

Theorem 2.1. *Assume that* $0 \leq \mathscr{d}_1 < \mathscr{d}_2$, $\Theta : \mathbb{I} = [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ *is a positive function and* $\Theta \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. *If* Θ *is a CF on* $\mathbb{I}$, *then*

$$\Theta\Big(\frac{\mathscr{d}_1 + \mathscr{d}_2}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(\mathscr{d}_2 - \mathscr{d}_1)^{\alpha}} [J^{\alpha}_{\mathscr{d}_1+}\Theta(\mathscr{d}_2) + J^{\alpha}_{\mathscr{d}_2-}\Theta(\mathscr{d}_1)] \leq \frac{\Theta(\mathscr{d}_1) + \Theta(\mathscr{d}_2)}{2}, \quad \alpha > 0.$$

In 2014, Set et al. [6] investigated a novel generalization of H-H inequality via R-LFIO involving s-CF.

Theorem 2.2. *Let $\alpha > 0$ and $s \in (0,1)$. Suppose $\circleddash : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ be a positive function with $0 \leq \mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. If $\circleddash$ is a s-CF in the 2^{nd} sense on $[\mathscr{d}_1, \mathscr{d}_2]$, then*

$$\begin{aligned} 2^{s-1}\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(\mathscr{d}_2-\mathscr{d}_1)^{\alpha}}[J^{\alpha}_{\mathscr{d}_1+}\circleddash(\mathscr{d}_2)+J^{\alpha}_{\mathscr{d}_2-}\circleddash(\mathscr{d}_1)] \\ &\leq \Big[\frac{1}{\alpha+s}+B(\alpha,s+1)\Big]\frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}. \end{aligned} \tag{3}$$

In 2014, Iscan et al. [7] investigated a novel generalization of H-H inequality via R-LFIO involving H-CF.

Theorem 2.3. *Assume that $\alpha > 0$, $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{I}^\circ$ with $\mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : \mathbb{I} \to \mathbb{R}$ is a function such that $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. If $\circleddash$ is H-CF on $\mathbb{I}$, then:*

$$\begin{aligned} \circleddash\Big(\frac{2x_1\mathscr{d}_2}{\mathscr{d}_1+\mathscr{d}_2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2}\Big(\frac{\mathscr{d}_1\mathscr{d}_2}{\mathscr{d}_2-\mathscr{d}_1}\Big)^{\alpha}\Big\{J^{\alpha}_{1/\mathscr{d}_1-}(\circleddash\circ g)(1/\mathscr{d}_2)+J^{\alpha}_{1/\mathscr{d}_2+}(\circleddash\circ g)(1/\mathscr{d}_1)\Big\} \\ &\leq \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}, \end{aligned}$$

with $g(x) = \frac{1}{x}, x \in \Big[\frac{1}{\mathscr{d}_2}, \frac{1}{\mathscr{d}_1}\Big].$

In 2015, Sarikaya et al. [8] investigated a novel generalization of H-H inequality via R-LFIO involving ϕ-CF.

Theorem 2.4. *Let $\mathscr{d}_1, \mathscr{d}_2 \in J$ with $\mathscr{d}_1 < \mathscr{d}_2$ and ϕ be increasing and continuous function. Suppose $\circleddash : \mathbb{I} \to \mathbb{R}$ be a ϕ-CF on $\mathbb{I} = [\mathscr{d}_1, \mathscr{d}_2]$, then:*

$$\begin{aligned} \circleddash\Big(\frac{\phi(\mathscr{d}_1)+\phi(\mathscr{d}_2)}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(\phi(\mathscr{d}_2)-\phi(\mathscr{d}_1))^{\alpha}}\Big[J^{a}_{\phi(\mathscr{d}_1)+}\circleddash(\phi(\mathscr{d}_2))+J^{a}_{\phi(\mathscr{d}_2)-}\circleddash(\phi(\mathscr{d}_1))\Big] \\ &\leq \frac{\circleddash(\phi(\mathscr{d}_1))+\circleddash(\phi(\mathscr{d}_2))}{2}. \end{aligned}$$

In 2016, Sarikaya et al. [9] investigated a novel generalization of H-H inequality via R-LFIO involving CF.

Theorem 2.5. *Assume that $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{R}$ with $0 \leq \mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : \mathbb{I} = [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is a positive function such that $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. If $\circleddash$ is a CF on $\mathbb{I}$, then:*

$$\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathscr{d}_2-\mathscr{d}_1)^{\alpha}}\Big[J^{\alpha}_{(\frac{\mathscr{d}_1+\mathscr{d}_2}{2})+}\circleddash(\mathscr{d}_2)+J^{\alpha}_{(\frac{\mathscr{d}_1+\mathscr{d}_2}{2})-}\circleddash(\mathscr{d}_1)\Big] \leq \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}, \quad \alpha > 0.$$

In 2018, Kunt et al. [10] investigated a novel generalization of H-H inequality via R-LFIO involving CF.

Theorem 2.6. *Assume that $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{R}$ with $\mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is a CF. Then:*

$$\circleddash\Big(\frac{\mathscr{d}_1+\alpha\mathscr{d}_2}{\alpha+1}\Big) \leq \frac{\Gamma(\alpha+1)}{(\mathscr{d}_2-\mathscr{d}_1)^{\alpha}}J^{\alpha}_{\mathscr{d}_2-}\circleddash(\mathscr{d}_1) \leq \frac{\alpha\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{\alpha+1}, \quad \alpha > 0.$$

In 2019, Kunt et al. [11] investigated a novel generalization of H-H inequality via R-LFIO involving CF.

Theorem 2.7. *Assume that $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{R}$ with $\mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is a convex function. If $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$, then:*

$$\circleddash\Big(\frac{\alpha\mathscr{d}_1+\mathscr{d}_2}{\alpha+1}\Big) \leq \frac{\Gamma(\alpha+1)}{(\mathscr{d}_2-\mathscr{d}_1)^{\alpha}}J^{\alpha}_{\mathscr{d}_1+}\circleddash(\mathscr{d}_2) \leq \frac{\alpha\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{\alpha+1}, \quad \alpha > 0.$$

In 2020, Budak et al. [12] investigated a novel generalization of H-H inequality via R-LFIO involving CF.

Theorem 2.8. *Assume that $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{R}$ with $\mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is positive, twice differentiable mapping, and $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. If $\circleddash'(\mathscr{d}_1+\mathscr{d}_2-x) \geq \circleddash'(x)$ for all $x \in \Big[\mathscr{d}_1, \frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big]$, then:*

$$\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathscr{d}_2-\mathscr{d}_1)^{\alpha}}\Big[J^{\alpha}_{(\frac{\mathscr{d}_1+\mathscr{d}_2}{2})+}\circleddash(\mathscr{d}_2)+J^{\alpha}_{(\frac{\mathscr{d}_1+\mathscr{d}_2}{2})-}\circleddash(\mathscr{d}_1)\Big] \leq \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}.$$

In 2020, Sanli et al. [13] investigated a novel generalization of H-H inequality via R-LFIO involving H-CF.

Theorem 2.9. *Assume that $\mathscr{d}_1, \mathscr{d}_2 \in \mathbb{I}^\circ$ with $\mathscr{d}_1 < \mathscr{d}_2$ and $\circleddash : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ is a H-CF such that $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. Then :*

$$\circleddash\Big(\frac{(\alpha+1)x_1x_2}{\mathscr{d}_1+\alpha\mathscr{d}_2}\Big) \le \Gamma(\alpha+1)\Big(\frac{\mathscr{d}_1\mathscr{d}_2}{\mathscr{d}_2-\mathscr{d}_1}\Big)^\alpha J^\alpha_{\frac{1}{\mathscr{d}_2}-}(\circleddash\circ h)\Big(\frac{1}{\mathscr{d}_1}\Big) \le \frac{\alpha\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{\alpha+1},$$

where $h(x) = \frac{1}{x}, x \in \Big[\frac{1}{\mathscr{d}_2}, \frac{1}{\mathscr{d}_1}\Big]$ and $\alpha > 0$.

In 2021, Barsam et al. [14] investigated a new form of H-H inequality via R-LFIO involving CF.

Theorem 2.10. *Assume that $\alpha > 0$ and $\circleddash : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is uniformly p-CF with ψ. Then :*

$$\begin{aligned}
&\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(\mathscr{d}_2-\mathscr{d}_1)^\alpha} J^\alpha_{(\mathscr{d}_1-\mathscr{d}_2)+}\psi(|\mathscr{d}_1-\mathscr{d}_2|)\\
\le\ & \frac{\Gamma(\alpha+1)}{(\mathscr{d}_2-\mathscr{d}_1)^\alpha}\Big[J^\alpha_{\mathscr{d}_1+}\circleddash(\mathscr{d}_2) + J^\alpha_{\mathscr{d}_2-}\circleddash(\mathscr{d}_1)\Big]\\
\le\ & 2(\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)) - 2\alpha B(\alpha+1,2)\circleddash(|\mathscr{d}_1-\mathscr{d}_2|).
\end{aligned}$$

2.1 H-H Type Inequality via Riemann-Liouville Fractional Intgerals of Functions w.r.t. to Another Function

Definition 3 ([3])**.** *Let $\psi : [\mathtt{u}_1, \mathtt{u}_2] \to \mathbb{R}$ be positive and increasing function on $(\mathtt{u}_1, \mathtt{u}_2]$, having ψ' on $(\mathtt{u}_1, \mathtt{u}_2)$. The R-LFIO w.r.t. the function ψ on $[\mathtt{u}_1, \mathtt{u}_2]$ of order $\alpha > 0$ are stated, by*

$$J^\alpha_{\mathtt{u}_1+;\psi}\circleddash(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{u}_1}^x [\psi(x)-\psi(v)]^{\alpha-1}\psi'(v)\circleddash(v)dv, \quad x > \mathtt{u}_1,$$

and

$$J^\alpha_{\mathtt{u}_2-;\psi}\circleddash(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\mathtt{u}_2} [\psi(v)-\psi(x)]^{\alpha-1}\psi'(v)\circleddash(v)dv, \quad x < \mathtt{u}_2,$$

provided that the integrals exists.

In 2016, Budak et al. [15] investigated a new construction of H-H inequality via R-LFIO w.r.t. another function involving s-CF.

Theorem 2.11. *Assume that $\psi : [\mathscr{d}_1, \mathscr{d}_2] \to \mathbb{R}$ is positive and increasing monotone function on $(\mathscr{d}_1, \mathscr{d}_2]$, having a continuous derivative on $(\mathscr{d}_1, \mathscr{d}_2)$ and let $\alpha > 0$. If $\circleddash$ is an s-CF in the 2^{nd} sense on $[\mathscr{d}_1, \mathscr{d}_2]$ for some fixed $s \in (0, 1]$, then:*

$$\begin{aligned}
& 2^{s-1}\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big)\\
\le\ & \frac{\Gamma(\alpha+1)}{2[\circleddash(\mathscr{d}_2)-\circleddash(\mathscr{d}_1)]^\alpha}\frac{1}{2}\big[I^{\alpha,\psi}_{\mathscr{d}_1+}[\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)] + I^{\alpha,\psi}_{\mathscr{d}_2-}[\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)]\big]\\
\le\ & \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}\frac{\alpha(\mathscr{d}_2-\mathscr{d}_1)}{[\circleddash(\mathscr{d}_2)-\circleddash(\mathscr{d}_1)]^\alpha}\frac{1}{2}\Big[\int_0^1 \frac{[t^s+(1-t)^s]\psi'((1-t)\mathscr{d}_1+tx_2)}{[\psi(\mathscr{d}_2)-\psi((1-t)\mathscr{d}_1+tx_2)]^{1-\alpha}}dt\\
& + \int_0^1 \frac{[t^s+(1-t)^s]\psi'((1-t)\mathscr{d}_1+tx_2)}{[\psi((1-t)\mathscr{d}_1+tx_2)-\psi(\mathscr{d}_1)]^{1-\alpha}}dt\Big].
\end{aligned}$$

In 2016, Jleli et al. [16] investigated a new extension of H-H inequality via R-LFIO w.r.t. another function involving CF.

Theorem 2.12. *Let $\alpha > 0$, $\circleddash$ be a CF on $[\mathscr{d}_1, \mathscr{d}_2]$ and $F(x) = \circleddash(x) + \circleddash(\mathscr{d}_1 + \mathscr{d}_2 - x)$. Then*

$$\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) \le \frac{\Gamma(\alpha+1)}{4(\psi(\mathscr{d}_2)-\psi(\mathscr{d}_1))^\alpha}\Big[I^{\alpha;\psi}_{\mathscr{d}_1+}F(\mathscr{d}_2) + I^{\alpha;\psi}_{\mathscr{d}_2-}F(\mathscr{d}_1)\Big] \le \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}.$$

In 2019, Liu et al. [17] investigated a new form of H-H inequality via R-LFIO w.r.t. another function involving CF.

Theorem 2.13. *Let $0 \le \mathscr{d}_1 < \mathscr{d}_2$, $\circleddash : \mathbb{I} \to \mathbb{R}$ be a positive function and $\circleddash \in L_1[\mathscr{d}_1, \mathscr{d}_2]$. In addition, suppose that $\circleddash$ is a CF on $\mathbb{I}$, having a continuous derivative ψ' on $\mathbb{I}$ and $\alpha \in (0, 1)$. Then:*

$$\begin{aligned}
\circleddash\Big(\frac{\mathscr{d}_1+\mathscr{d}_2}{2}\Big) &\le \frac{\Gamma(\alpha+1)}{2(\mathscr{d}_2-\mathscr{d}_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(\mathscr{d}_1)+}(\circleddash\circ\psi)(\psi^{-1}(\mathscr{d}_2)) + I^{\alpha;\psi}_{\psi^{-1}(\mathscr{d}_2)-}(\circleddash\circ\psi)(\psi^{-1}(\mathscr{d}_1))\Big]\\
&\le \frac{\circleddash(\mathscr{d}_1)+\circleddash(\mathscr{d}_2)}{2}.
\end{aligned}$$

2.2 H-H Type Inequality via k-Riemann-Liouville Fractional Intgeral Operator

Definition 4 ([18])**.** *Let $\circleddash \in L[\mathtt{u}_1, \mathtt{u}_2]$, $a \geq 0$, and $k > 0$. The k-RLFIO for $\alpha > 0$ are defined by*

$$I^{\alpha}_{\mathtt{u}_1+,k}\circleddash(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{\mathtt{u}_1}^{t} (t-s)^{\frac{\alpha}{k}-1}\circleddash(s)ds, \quad t > \mathtt{u}_1,$$

and

$$I^{\alpha}_{\mathtt{u}_2-,k}\circleddash(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{t}^{\mathtt{u}_2} (s-t)^{\frac{\alpha}{k}-1}\circleddash(s)ds, \quad t < \mathtt{u}_2,$$

respectively.

In 2021, Wu et al. [19] investigated a new kind of H-H inequality via k-R-LFIO involving CF.

Theorem 2.14. *Assume that $\circleddash : \mathbb{I} \to \mathbb{R}$ is a positive function with $0 \leq \ell_1 < \ell_2$, $\circleddash \in L_1[\ell_1, \ell_2]$. If $\circleddash$ is a CF on $\mathbb{I}$, then*

$$\circleddash\Big(\frac{\ell_1+\ell_2}{2}\Big) \leq \frac{\Gamma_k(\alpha+k)}{2(\ell_2-\ell_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\ell_1+,k}\circleddash(\ell_2) + I^{\alpha}_{\ell_2-,k}\circleddash(\ell_1)\Big] \leq \frac{\circleddash(\ell_1)+\circleddash(\ell_2)}{2}.$$

Theorem 2.15. *Assume that $k > 0$, and $\circleddash : \mathbb{I} \to \mathbb{R}$ is a positive mapping with $0 \leq \ell_1 < \ell_2$, $\circleddash \in L_1[\ell_1, \ell_2]$. If $\circleddash$ is a CF on $\mathbb{I}$, then*

$$\circleddash\Big(\frac{\ell_1+\ell_2}{2}\Big) \leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\ell_2-\ell_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{(\frac{\ell_1+\ell_2}{2})+,k}\circleddash(\ell_2) + I^{\alpha}_{(\frac{\ell_1+\ell_2}{2})-,k}\circleddash(\ell_1)\Big] \leq \frac{\circleddash(\ell_1)+\circleddash(\ell_2)}{2}.$$

In 2021, Sahoo et al. [20] investigated a novel kind of H-H inequality via k-R-LFIO involving h-CF.

Theorem 2.16. *Let a function $\circleddash : \mathbb{I} \to \mathbb{R}$ be h-CF with $0 \leq \ell_1 \leq \ell_2$. If $\circleddash \in L_1[\ell_1, \ell_2]$, then:*

$$\begin{aligned} &\frac{\Gamma_k(\alpha+k)}{(\ell_2-\ell_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\ell_1+,k}\circleddash(\ell_2) + I^{\alpha}_{\ell_2-,k}\circleddash(\ell_1)\Big] \leq \frac{\alpha[\circleddash(\ell_1)+\circleddash(\ell_2)]}{k}\int_0^1 s^{\frac{\alpha}{k}}[h(s)+h(1-s)]ds \\ \leq\; &\frac{\alpha[\circleddash(\ell_1)+\circleddash(\ell_2)]}{k^{\frac{p-1}{p}}}\Big(\frac{1}{p\alpha-pk+k}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^1 (h(1-s))^r ds\Big)^{\frac{1}{r}} + \Big(\int_0^1 h(s)^r ds\Big)^{\frac{1}{r}}\Big], \end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

3. CONCLUSIONS

This review paper's goal was to present an in depth and fresh analysis of H-H inequalities related to R-L fractional operators. The theoretical and practical implications of the H-H-type inequalities was taken in to account when preparing this review. We think that this current overview will inspire and give scholars studying H-H inequalities a place to learn about prior studies on the matter before forthcoming up with new conclusions. The review's consequences for future study are encouraging, and we expect it to stimulate a great deal more research.

Conflicts of Interest

The authors declare no conflicts of interest.

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