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Research Article

Some Hermite-Hadamard Type Integral Inequalities and Their Applications via the Modified Riemann-Liouville Fractional Integral Operator

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ABSTRACT

The aim of this manuscript is to introduce a novel form of H-H inequality via ψ-RLFIO for preinvex functions. By employing this approach, we construct a new lemma. In addition, based on this newly derived fractional identity, some new estimation of fractional H-H inequality involving m-preinvex via ψ-RLFIO is investigated. Further, we add some mean-type applications.

1. INTRODUCTION

Convex functions have a long and illustrious history. The background of convexity theory can be traced all the way back to the end of the nineteenth century. Convex theory provides us with appropriate guidelines and techniques to focus on a broad range of problems in applied sciences. It has been widely acknowledged in recent years that mathematical inequalities have contributed to the development of various aspects of mathematics as well as other scientific disciplines. The term invex function first time examined by Hanson [1]. Mond and Weir [2] explored the notion of preinvexity. The analysis of the preinvex and invex theory utilizing the bifunction by Mond and Ben-Israel [3] can be discussed as meaningful addition to the optimization field.
The aim and novelty of this work are to introduce a novel form of H-H type integral inequality via preinvexity in the frame of $\Psi - RLFIO$. Further, we are to construct some refinements of H-H type integral inequality via Ψ-RLFIO..

2. PRELIMINARIES

In this part, we remember some basic concepts required for this manuscript.

Definition 2.1 ([4]). *Let $\mathcal{A} \subset \mathbb{R}^n$ be invex with respect to $\Pi(.,.)$, if*

$$\mathfrak{r}_1 + \delta\Pi(\mathfrak{r}_2, \mathfrak{r}_1) \in \mathcal{A},$$

$\forall\ \mathfrak{r}_1, \mathfrak{r}_2 \in \mathcal{A}$ *and* $\delta \in [0, 1]$.

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Definition 2.2 ([5]). *Let $\Pi : \mathcal{A} \times \mathcal{A} \times (0,1] \to \mathbb{R}^n$ and $\mathcal{A} \subseteq \mathbb{R}^n$. Then $\mathcal{A}$ is m–invex with respect to Π, if*

$$m\tau_2 + \delta\Pi(\tau_1, \tau_2, m) \in \mathcal{A}$$

holds $\forall\ \tau_1, \tau_2 \in \mathcal{A}$, $m \in (0,1]$ and $\delta \in [0,1]$.

Example 2.1 ([5]). *Suppose $m = \frac{1}{4}$, $\mathcal{A} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ and*

$$\Pi(\tau_2, \tau_1, m) = \begin{cases} m\cos(\tau_2 - \tau_1) & if\ \tau_1 \in (0, \frac{\pi}{2}], \tau_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\tau_2 - \mu_1) & if\ \tau_1 \in [\frac{-\pi}{2}, 0), \tau_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\tau_1) & if\ \tau_1 \in (0, \frac{\pi}{2}], \tau_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\tau_1) & if\ \tau_1 \in [-\frac{\pi}{2}, 0), \tau_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

Then, $\mathcal{A}$ is an m–invex set but not convex $\forall\ \wp \in [0,1]$.

Definition 2.3 ([2]). *Let $\Pi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^n$ and $\mathcal{A} \subseteq \mathbb{R}^n$. Then $\circleddash : \mathcal{A} \to \mathbb{R}$ is preinvex with respect to Π if*

$$\circleddash\left(\tau_2 + \wp\Pi\left(\tau_1, \tau_2\right)\right) \leq \wp\circleddash\left(\tau_1\right) + (1-\wp)\circleddash\left(\tau_2\right), \qquad \forall \tau_1,\ \tau_2 \in \mathcal{A}, \quad \wp \in [0,1].$$

Definition 2.4. *[6] Let $\Pi : \mathcal{A} \times \mathcal{A} \times (0,1] \to \mathbb{R}^n$ and $\mathcal{A} \subseteq \mathbb{R}^n$. Then $\circleddash : \mathcal{A} \to \mathbb{R}$ is generalized m-preinvex with respect to Π if*

$$\circleddash(m\tau_2 + \wp\Pi(\tau_1, \tau_2, m)) \leq \wp\circleddash\left(\tau_1\right) + m\left(1-\wp\right)\circleddash\left(\tau_2\right), \tag{1}$$

holds for every $\tau_1, \tau_2 \in \mathcal{A}$, $m \in (0,1]$ and $\wp \in [0,1]$.

Condition C: Let $\mathcal{A} \subset \mathbb{R}$ is an open invex subset with respect to $\Pi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$. We say that the function Π satisfied the condition C, if for any $\tau_1, \tau_2 \in \mathcal{A}$ and $\wp \in [0,1]$,

$$\begin{aligned} \Pi\left(\tau_1, \tau_1 + \wp\,\Pi\left(\tau_2, \tau_1\right)\right) &= -\wp\,\Pi\left(\tau_2, \tau_1\right) \\ \Pi\left(\tau_2, \tau_1 + \wp\,\Pi\left(\tau_2, \tau_1\right)\right) &= (1-\wp)\ \Pi\left(\tau_2, \tau_1\right). \end{aligned}$$

For any $\tau_1, \tau_2 \in \mathcal{A}$, $\wp_1, \wp_2 \in [0,1]$, then according to the above equations, we have

$$\Pi\left(\tau_1 + \wp_2\,\Pi\left(\tau_2, \tau_1\right),\ \tau_1 + \wp_1\,\Pi\left(\tau_2, \tau_1\right)\right) = (\wp_2 - \wp_1)\,\Pi\left(\tau_2, \tau_1\right).$$

This Condition is very important in the optimization and creation of the theory of inequalities (see [7, 8]).
The following extended Condition C in the frame of m-preinvexity was also discussed by Du et.al in [9].
Extended Condition C: Assume that $\mathcal{A} \subset \mathbb{R}$ an open invex subset w.r.t. $\Pi : \mathcal{A} \times \mathcal{A} \times (0,1] \to \mathbb{R}$. We say that the function Π satisfied the Extended Condition C, if for any $\tau_1, \tau_2 \in \mathcal{A}$, $\wp \in [0,1]$, we have

$$\begin{aligned} \Pi\left(\tau_2, m\tau_2 + \wp\,\Pi\left(\tau_1, \tau_2, m\right), m\right) &= -\wp\,\Pi\left(\tau_1, \tau_2, m\right) \\ \Pi\left(\tau_1, m\tau_2 + \wp\,\Pi\left(\tau_1, \tau_2, m\right), m\right) &= (1-\wp)\ \Pi\left(\tau_1, \tau_2, m\right) \\ \Pi\left(\tau_1, \tau_2, m\right) &= -\Pi\left(\tau_2, \tau_1, m\right). \end{aligned}$$

Theorem 2.1. *[10] Assume that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume thats $\Psi_1, \Psi_2 : [x_1, x_2] \to \mathbb{R}$ are such that $|\Psi_1|^p$ and $|\Psi_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_0^1 |\circleddash_1(x)\circleddash_2(x)|dx \leq \Big(\int_0^1 |\circleddash_1(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_0^1 |\circleddash_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

Definition 2.5. *[11] Let $(\tau_1, \tau_2)(-\infty \leq \tau_1 < \tau_2 \leq \infty)$ be an interval of the real line R and $\alpha > 0$. Also let $\psi(w)$ be positive monotone and increasing function on $(\tau_1, \tau_2]$, having a continuous derivative $\psi'(w)$ on $(\tau_1, \tau_2])$. Then ψ-R-LFIO is given by:*

$$I_{\tau_1^+}^{\alpha:\psi}\circleddash(w) = \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{w} \psi'(\mu)(\psi(w) - \psi(\mu))^{\alpha-1}\circleddash(\mu)d\mu,$$

$$I_{\tau_1^-}^{\alpha:\psi}\circleddash(w) = \frac{1}{\Gamma(\alpha)}\int_{w}^{\tau_2} \psi'(\mu)(\psi(\mu) - \psi(w))^{\alpha-1}\circleddash(\mu)d\mu,$$

respectively.

3. H-H INEQUALITY VIA Ψ-RLFIO

Here, we investigate a novel form of H–H-type inequality for a m-preinvex function via Ψ-RLFIO.

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Pi : I \times I \to \mathbb{R}$ and $\tau_1, \tau_2 \in I$ with $m\tau_1 < m\tau_1 + \Pi(\tau_2, \tau_1, m)$. If $\Theta : [m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)] \to \mathbb{R}$ is a m-preinvex function and $\Theta \in L[m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)]$ and Π satisfies extended condition C. Also suppose $\Psi(w)$ be positive function and increasing on $(m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)$, having a continous drivative $\Psi(w)'$ on $(m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)$ and $\alpha \in (0, 1)$, then*

$$\begin{aligned}
&\Theta(m\tau_1 + \frac{1}{2}\Pi(\tau_2, \tau_1, m)) \\
&\leq \frac{\Gamma(\alpha + 1)}{2\Pi^{\alpha}(\tau_2, \tau_1, m)}[I^{\alpha:\psi}_{\psi^{-}(m\tau_1)^{+}}(\Theta o\psi)\psi^{-1}(m\tau_1 + \Pi(\tau_2, \tau_1, m))] \\
&+ [I^{\alpha:\psi}_{\psi^{-}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\Theta o\psi)\psi^{-1}(m\tau_1)] \\
&\leq \frac{\Theta(m\tau_1) + \Theta(m\tau_1 + \Pi(\tau_2, \tau_1, m)}{2} \leq \frac{\Theta(m\tau_1) + \Theta(\tau_2)}{2}.
\end{aligned} \tag{2}$$

Proof. Since Θ is m-preinvex on $[m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)]$, we can write

$$\Theta(mw + \frac{1}{2}\Pi(z, w, m)) \leq \frac{\Theta(mw) + \Theta(z)}{2}.$$

Using $w = m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m)$ and $z = m\tau_1 + \delta\Pi(\tau_2, \tau_1, m)$ in (2), we have

$$\begin{aligned}
&\Theta(m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m) + \frac{1}{2}\Pi(m\tau_1 + \delta\Pi(\tau_2, \tau_1, m), m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m))) \\
&\leq \frac{\Theta(m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m)) + \Theta(m\tau_1 + \delta\Pi(\tau_2, \tau_1, m)}{2}.
\end{aligned} \tag{3}$$

Employing extended Condition C in (3), we have

$$\begin{aligned}
&\Theta(m\tau_1 + \frac{1}{2}\Pi(\tau_2, \tau_1, m)) \\
&\leq \frac{\Theta(m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m)) + \Theta(m\tau_1 + \delta\Pi(\tau_2, \tau_1, m))}{2}.
\end{aligned} \tag{4}$$

Multiplying inequality (4) by $\delta^{\alpha-1}$ then integrating over $[0, 1]$, we attain

$$\begin{aligned}
&\frac{2}{\alpha}\Theta(m\tau_1 + \frac{1}{2}\Pi(\tau_2, \tau_1, m)) \\
&\leq \int_0^1 \delta^{\alpha-1}\Theta(m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m))d\delta + \int_0^1 \delta^{\alpha-1}\Theta(m\tau_1 + \delta\Pi(\tau_2, \tau_1, m))d\delta.
\end{aligned}$$

Next

$$\begin{aligned}
&\frac{\Gamma(\alpha + 1)}{2\Pi^{\alpha}(\tau_2, \tau_1, m)}[I^{\alpha:\psi}_{\psi^{-}(m\tau_1)^{+}}(\Theta o\psi)\psi^{-1}(m\tau_1 + \Pi(\tau_2, \tau_1, m))] + [I^{\alpha:\psi}_{\psi^{-}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\Theta o\psi)\psi^{-1}(m\tau_1)] \\
&= \frac{\alpha}{2\Pi^{\alpha}(\tau_2, \tau_1, m)}[\int_{\psi^{-1}(m\tau_1)}^{\psi^{-1}(m\tau_1+\Pi^{\alpha}(\tau_2,\tau_1,m)} (m\tau_1 + \Pi(\tau_2, \tau_1, m) - \psi(\mu))^{\alpha-1}(\Theta o\psi)(\mu)\psi'(\mu)d\mu \\
&+ \int_{\psi^{-1}(m\tau_1)}^{\psi^{-1}(m\tau_1+\Pi^{\alpha}(\tau_2,\tau_1,m)} (\psi(\mu) - m\tau_1)^{\alpha-1}(\Theta o\psi)(\mu)\psi'(\mu)d\mu] \\
&= \frac{\alpha}{2}\int_0^1 \delta^{\alpha-1}\Theta(m\tau_1 + (1 - \delta)\Pi(\tau_2, \tau_1, m))d\delta + \int_0^1 \delta^{\alpha-1}\Theta(m\tau_1 + \delta\Pi(\tau_2, \tau_1, m))d\delta.
\end{aligned} \tag{5}$$

From the inequalities (4) and (5), we get

$$\begin{aligned}
&\Theta(m\tau_1 + \frac{1}{2}\Pi(\tau_2, \tau_1, m)) \leq \frac{\Gamma(\alpha + 1)}{2\Pi^{\alpha}(\tau_2, \tau_1, m)}[I^{\alpha:\psi}_{\psi^{-}(m\tau_1)^{+}}(\Theta o\psi)\psi^{-1}(m\tau_1 + \Pi(\tau_2, \tau_1, m))] \\
&+ [I^{\alpha:\psi}_{\psi^{-}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\Theta o\psi)\psi^{-1}(m\tau_1)].
\end{aligned} \tag{6}$$

For the second inequality, we have

$$\begin{aligned}&\mho(m\tau_1+\delta\Pi(\tau_2,\tau_1,m))=\mho(m\tau_1+\Pi(\tau_2,\tau_1,m)+(1-\delta)\Pi(m\tau_1,m\tau_1+\delta\Pi(\tau_2,\tau_1,m)))\\&\leq\mho(m\tau_1+\Pi(\tau_2,\tau_1,m)+(1-\delta)\mho(m\tau_1).\end{aligned}\tag{7}$$

and

$$\begin{aligned}&\mho(m\tau_1+(1-\delta)\Pi(\tau_2,\tau_1,m))=\mho(m\tau_1+\Pi(\tau_2,\tau_1,m)+\delta\Pi(m\tau_1,m\tau_1+\Pi(\tau_2,\tau_1,m)))\\&\leq(1-\delta)\mho(m\tau_1+\Pi(\tau_2,\tau_1,m))+\delta\mho(m\tau_1).\end{aligned}\tag{8}$$

From the inequalities (7) and (8), we get

$$\begin{aligned}&\mho(m\tau_1+\Pi(\tau_2,\tau_1,m))+\mho(m\tau_1+(1-\delta)\Pi(\tau_2,\tau_1,m))+\delta\mho(m\tau_1))\\&\leq\mho(m\tau_1)+\mho(m\tau_1+\Pi(\tau_2,\tau_1,m)).\end{aligned}\tag{9}$$

Then, multiplying inequality (9) by $\delta^{\alpha-1}$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\delta^{\alpha-1}\mho(m\tau_1+\delta\Pi(\tau_2,\tau_1,m))d\delta+\int_0^1\delta^{\alpha-1}\mho(m\tau_1+(1-\delta)\Pi(\tau_2,\tau_1,m))d\delta\\&\leq\frac{\mho(m\tau_1)+\mho(m\tau_1+\Pi(\tau_2,\tau_1,m)}{\alpha}.\end{aligned}\tag{10}$$

From the inequalities (6) and (10), we get

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}[I^{\alpha:\psi}_{\psi^{-}(m\tau_1)^{+}}(\mho o\psi)\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))\\&+I^{\alpha:\psi}_{\psi^{-}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\mho o\psi)\psi^{-1}(m\tau_1)]\\&\leq\frac{\mho(m\tau_1)+\mho(m\tau_1+\Pi(\tau_2,\tau_1,m))}{\alpha}\leq\frac{\mho(m\tau_1)+\mho(\tau_2)}{2}.\end{aligned}$$

This completes the proof.

4. GENERALIZATION OF H–H-TYPE INEQUALITY VIA Ψ-RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL OPERATOR

Lemma 4.1. *Let I and Π are defined in Theorem 3.1. Suppose that $\Psi : I \to \mathbb{R}$ be a differentiable function. If Ψ' is m-preinvx and $\Psi' \in L[m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)]$. Also suppose $\Psi(w)$ is a positive monotone and increasing function on $(m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)$, having a continous drivative $\Psi(w)'$ on $(m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)$ and $\alpha\in(0,1)$, then*

$$\begin{aligned}&\frac{\mho(m\tau_1)+\mho(m\tau_1+\Pi(\tau_2,\tau_1,m))}{\alpha}-\frac{\Gamma(\alpha+1)}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}\Big[I^{\alpha:\psi}_{\psi^{-}(m\tau_1)^{+}}(\mho o\psi)\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))\\&+I^{\alpha:\psi}_{\psi^{-}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\mho o\psi)\psi^{-1}(m\tau_1)\Big]\\&=\frac{1}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}\int_{\psi^{-1}(m\tau_1)}^{\psi^{-1}(m\tau_1+\Pi^{\alpha}(\tau_2,\tau_1,m)}[(\psi(\mu)-m\tau_1)^{\alpha}-(m\tau_1+\Pi(\tau_2,\tau_1,m)-\psi(\mu))^{\alpha}](\mho' o\psi)(\mu)\psi'(\mu)d\mu\\&=\frac{\Pi(\tau_2,\tau_1,m)}{2}\int_0^1((1-\delta)^{\alpha}-\delta^{\alpha})\mho'(m\tau_1+(1-\delta)\Pi(\tau_2,\tau_1,m))d\delta,\end{aligned}$$

where $\alpha\in(0,1]$, $\delta\in[0,1]$.

Theorem 4.1. *Let I and Π are defined in Theorem 3.1. Suppose that $\Psi : I \to \mathbb{R}$ be a differentiable function. If Ψ' is m-preinvx and $\Psi' \in L[m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)]$. Also suppose $\Psi(w)$ is a positive monotone and increasing function on $(m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)$, having a continous drivative $\Psi(w)'$ on $(m\tau_1, m\tau_1+\Pi(\tau_2,\tau_1,m)$ and $\alpha\in(0,1)$, then*

$$\begin{aligned}&\Bigg|\frac{\mho(m\tau_1)+\mho(m\tau_1+\Pi(\tau_2,\tau_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}\Big[I^{\alpha:\psi}_{\psi^{-1}(m\tau_1)^{-}}(\mho o\psi)\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))\\&+I^{\alpha:\psi}_{\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))^{-}}(\mho o\psi)(m\tau_1)\Big]\Bigg|\leq\frac{\Pi(\tau_2,\tau_1,m)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|\mho'(m\tau_1)|^q+|\mho'(\tau_2)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $q>1$, $\alpha\in(0,1]$.

Proof. By using Lemma 4.1, we get

$$\left|\frac{\mathcal{O}(m\tau_1)+\mathcal{O}(m\tau_1+\Pi(\tau_2,\tau_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}\left[I^{\alpha:\psi}_{\psi^{-1}(m\tau_1)^-}(\mathcal{O}o\psi)\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))\right.\right.$$
$$\left.\left.+I^{\alpha:\psi}_{\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))^-}(\mathcal{O}o\psi)(m\tau_1)\right]\right|$$
$$\leq\frac{\Pi(\tau_2,\tau_1,m)}{2}\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}||\mathcal{O}'(m\tau_1+\delta\Pi(\tau_2,\tau_1,m))|d\delta.$$

By applying Hölder inequality, we get

$$\left|\frac{\mathcal{O}(m\tau_1)+\mathcal{O}(m\tau_1+\Pi(\tau_2,\tau_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Pi^{\alpha}(\tau_2,\tau_1,m)}\left[I^{\alpha:\psi}_{\psi^{-1}(m\tau_1)^-}(\mathcal{O}o\psi)\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))\right.\right.$$
$$\left.\left.+I^{\alpha:\psi}_{\psi^{-1}(m\tau_1+\Pi(\tau_2,\tau_1,m))^-}(\mathcal{O}o\psi)(m\tau_1)\right]\right|$$
$$\leq\frac{\Pi(\tau_2,\tau_1,m)}{2}\left(\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}|^p\,d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{O}'(m\tau_1+\delta\Pi(\tau_2,\tau_1,m))\right|^q d\delta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Pi(\tau_2,\tau_1,m)}{2}\left(\int_0^1|(1-2\delta)|^{\alpha p}\,d\delta\right)^{\frac{1}{p}}\left(\int_0^1((1-\delta)\left|\mathcal{O}'(m\tau_1)\right|^q+\delta\left|\mathcal{O}'(\tau_2)\right|^q)\right)^{\frac{1}{q}}$$
$$=\frac{\Pi(\tau_2,\tau_1,m)}{2(\alpha p+1)}^{\frac{1}{p}}\left(\frac{|\mathcal{O}'(m\tau_1)|^q+|\mathcal{O}'(\tau_2)|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof.

5. APPLICATION TO SPECIAL MEANS:

We recall the following means for two real numbers $\tau_1,\tau_2,\tau_1\neq\tau_2$.

$$A(\tau_1,\tau_2)=\frac{\tau_1+\tau_2}{2},\tau_1,\tau_2\in R,$$

$$H(\tau_1,\tau_2)=\frac{2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}},\tau_1,\tau_2\in R\backslash\{0\},$$

$$L(\tau_1,\tau_2)=\frac{\tau_2-\tau_1}{ln|\tau_2|-ln|\tau_1|},|\tau_1|\neq|\tau_2|,\tau_1\tau_2\in R,\tau_1\tau_2\neq 0,$$

$$L_n(\tau_1,\tau_2)=\left[\frac{\tau_2^{n+1}+\tau_1^{n+1}}{(n+1)(\tau_2-\tau_1)}\right]^{\frac{1}{n}},n\in Z\backslash\{-1,0\},\tau_1,\tau_2\in R,\tau_1\neq\tau_2.$$

Proposition 5.1. *Let $m\tau_1,m\tau_1+\Pi(\tau_2,\tau_1,m)\in R^+,m\tau_1<m\tau_1+\Pi(\tau_2,\tau_1,m)$. Then*

$$\left|A(m\tau_1^n,(m\tau_1+\Pi(\tau_2,\tau_1,m))^n)-L_n^n(m\tau_1,(m\tau_1+\Pi(\tau_2,\tau_1,m))\right|\leq\frac{n\Pi(\tau_2,\tau_1,m)}{2(p+1)^{\frac{1}{p}}}\left(\frac{m\tau_1^{(n-1)q}-\tau_2^{(n-1)q}}{2}\right)^{\frac{1}{q}}.$$

Proof. If $\alpha=m=1$, $\Psi(w)=w$ and $\mathcal{O}(w)=w^n$, in Theorem 4.1, then we derived the result.

Proposition 5.2. *Let $m\tau_1,m\tau_1+\Pi(\tau_2,\tau_1,m)\in R^+,m\tau_1<m\tau_1+\Pi(\tau_2,\tau_1,m)$. Then*

$$|A(e^{m\tau_1},e^{m\tau_1+\Pi(\tau_2,\tau_1,m)})-L(e^{m\tau_1},e^{m\tau_1+\Pi(\tau_2,\tau_1,m)})|\leq\frac{\Pi(\tau_2,\tau_1,m)}{2(p+1)^{\frac{1}{p}}}\left(\frac{e^{\tau_1 q}+e^{\tau_2 q}}{2}\right)^{\frac{1}{q}}.$$

Proof. If $\alpha = m = 1$, $\Psi(w) = w$, and $\circleddash(w) = e^w$ in Theorem 4.1, then we derived the result.

Proposition 5.3. *Let* $m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m) \in R^+, m\tau_1 < m\tau_1 + \Pi(\tau_2, \tau_1, m)$. *Then*

$$\left|H^{-1}(m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m)) - L^{-1}(m\tau_1, m\tau_1 + \Pi(\tau_2, \tau_1, m))\right| \leq \frac{\Pi(\tau_2, \tau_1, m)}{2(p+1)^{\frac{1}{p}}}\left[\frac{1}{2}\left(\frac{1}{\tau_1^{2q}} + \frac{1}{\tau_2^{2q}}\right)\right].$$

Proof. If $\alpha = m = 1$, $\psi(w) = w$ and $\circleddash(w) = \frac{1}{w}$, in Theorem 4.1, then we derived the result.

6. CONCLUSIONS

Fractional calculus has sparked the interest of multiple authors as well as scholars from a wide range of fields. Convexity theory allows us to create new, innovative numerical model frameworks that may be utilized to tackle a wide range of problems in the applied and pure sciences. Thus, convex analysis and its associated inequalities are growing in academic attention and appeal due to several advancements, modifications, and uses.
In this work:

(1) We investigated a new sort of H–H inequality via Ψ-RLFIO.

(2) We introduced a new lemma. Further, we discussed a new refinements of the H–H inequality based on newly constructed lemma.

(3) We introduced mean type applications in the frame of the Ψ-RLFIO .

Conflicts Of Interest

The authors declare no conflicts of interest.

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