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Research Article

Novel Homomorphic Characteristics of PU-Algebra in Relation to Discrete Dynamical Systems

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ABSTRACT

PU-algebra theory plays a pivotal role in various applied domains, including information sciences, cybernetics, computer science, and artificial intelligence. This work advances the theoretical understanding of PU-algebras, with a particular emphasis on their homomorphic properties. We explore foundational properties of PU-algebras, focusing on characterizing ideals and subalgebras by introducing discrete dynamical system concepts for instance periodic points, fixed points, invariant sets, and strongly invariant sets within the PU-algebraic framework. Here, $(\boldsymbol{\chi}, \boldsymbol{\psi})$ denotes a discrete dynamical system where $\boldsymbol{\chi}$ is a PU-algebra and $\boldsymbol{\psi}$ a homomorphism on $\boldsymbol{\chi}$. The findings not only deepen the theoretical foundation of PU-algebras but also pave the way for future applications in modeling complex dynamical systems and automata theory. This research opens several avenues for further study, including the development of algorithms for computational representation of PU-algebraic structures and the exploration of their utility in enhancing machine learning models and artificial intelligence systems. Additionally, gaps in understanding the interaction between PU-algebra properties and advanced dynamical system concepts suggest a promising direction for future investigation.

1. INTRODUCTION

Mostafa et al. laid the groundwork for the notion of PU-Algebra in their seminal paper [1]. Their PU-algebra theory, as well as Similar concepts and attributes are currently often used in areas such as information sciences, cybernetics, computer sciences, and artificial intelligence. Two considerations, one focused on Meredith's classical and non-classical propositional calculi [2] and set theory had an impact on algebras such as PU-algebras, BCK-algebras, and BCI-algebras. Mostafa et al. [1], first introduced the definition of ideal theory of PU-algebra and they also established PU-homomorphism. Several scholars have researched this algebra in depth throughout the last five years, leading to a substantial amount of work on PU-algebra theory [3-8].

In his popular paper celestial mechanics [9], the eminent mathematician Henri Poincare laid the foundation for dynamical systems. His observations regarding dynamical systems, as well as related ideas and properties, are now commonly used in physics, biology, meteorology, astronomy, economics, and other fields of research. The key idea of dynamical systems research is to describe the eventual or asymptotic action of an iterative mechanism using mathematical techniques in various scientific disciplines. Gaston Julia and Pierre Fatau [10] diverged the idea of dynamical system theory in relation to complex analysis in 1917, establishing a new concept and naming it complex analytic dynamical system. Later, Birkhoff [11] vigorously followed Poincare's perspective, recognizing the importance of the principle of mappings and introducing discrete dynamical systems. Discrete dynamical systems are a fascinating and active field of pure and applied mathematics that incorporates techniques and methods from a variety of disciplines, including as analysis, number theory, and geometry..

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Discrete dynamical system is dynamical system whose state evolves in discrete steps pursuant to a fixed rule over a state space. Birkhoff's discrete-step opinion was enthralling and it drew the attention of mathematicians who wanted to apply it to different disciplines of mathematics. Most of the researchers have used the principle of this system in their study. In certain fields of mathematics, this turn out to be an expansion of the theory of discrete dynamical systems. Birkhoff [12] introduced the concept of discrete dynamical systems to topology in 1927, laying the groundwork for a new field known as topological dynamics. Topological dynamics grew in popularity leading to the development of algebraic and differential topology [13]. Piexoto and Smale used differential topological techniques to examine the chaotic dynamics of an array of dynamical structures, and they established new types of dynamical systems called hyperbolic dynamical systems [14]. In measure theory, Von Neumann et al. [15], presented discrete dynamical systems, which gave rise to a new field known as Ergodic theory. Dikranjan and Bruno [16] developed another algebraic structure called discrete dynamical system in group theory by inserting the idea of discrete dynamical systems in group theory. Dawood et al. [17] recently established the notion of a discrete dynamical system into BCI-algebra and observed some intriguing properties. For further details, related to discrete dynamical system, ordered structural algebras and direct product see [18].

2. PRELIMINARIES

This section includes some basic facts concerning PU-algebra and some preliminary definitions that are significant in establishing our key claims. In this study, the PU-algebra is denoted by χ without any further explanation.
In this context, we only discuss PU-algebra tenets that are crucial to our approach. For further details on PU-algebra, readers may consult [1-8].

Definition 2.1: PU-algebras $(\chi, *, 0)$ represent a distinct category of algebra (of (2,0)-type) that conform to the following assumptions for any $u_0, v_0, w_0 \in \chi$, where

(P1) $0 * u_0 = 0$ **(P2)** $(u_0 * w_0) * (v_0 * w_0) = v_0 * u_0$

Proposition 2.2:(see [1]) The following PU-algebra χ criteria are met:

(C-1) The right cancellation law holds in χ
i.e., If $u_0 * v_0 = w_0 * v_0$ then $u_0 = w_0, \forall u_0, v_0, w_0 \in \chi$

(C-2) The left cancellation legislation is applicable in χ
i.e., If $v_0 * u_0 = v_0 * w_0$ then $u_0 = w_0, \forall u_0, v_0, w_0 \in \chi$

(C-3) $u_0 * u_0 = 0, \quad \forall u_0 \in \chi$

Definition 2.3:[1] Let $\chi_s \neq \{\}$ and is a subset of $(\chi, *, 0)$ then χ_s is termed a PU-subalgebra of χ, if $u_0 * v_0 \in \chi_s$ whenever $u_0, v_0 \in \chi_s$.

Definition 2.4:[1]Let $I_{\mathfrak{h}} \neq \{\}$and $I_{\mathfrak{h}} \subseteq \chi$where χ is a PU-algebra then $I_{\mathfrak{h}}$ is said to be an ideal of χ, if it fulfills conditions ($I_{\mathfrak{h}}$-1) and ($I_{\mathfrak{h}}$-2), where

($I_{\mathfrak{h}}$-1) $0 \in I_{\mathfrak{h}}$

($I_{\mathfrak{h}}$-2) $u_0 * v_0 \in I_{\mathfrak{h}}$ and $u_0 \in I_{\mathfrak{h}} \Rightarrow v_0 \in I_{\mathfrak{h}}$ for any $u_0, v_0 \in \chi$

Definition 2.5:[1] A mapping $\psi: \chi_1 \longrightarrow \chi_2$, where χ_1 and $\chi_2$are two PU-algebras meet the following criteria, they are said to be PU-homeomorphisms.

$\psi(u_0 * v_0) = \psi(u_0) * \psi(v_0) \quad , \forall u_0, v_0 \in \chi_1$

3. A New Perspective on Homomorphic Properties in PU-Algebra Using Certain Concepts of Discrete Dynamical Systems

Some terms are chosen in this section say periodic point, fixed point, invariant and (S-invariant) or strongly invariant sets of discrete dynamical systems and define them with respect to PU-algebras and then obtain some new homomorphic properties and theorems related to them.

Definition 3.1: Let χ be a PU-algebra and $\psi: \chi \longrightarrow \chi$ be a homomorphism then, in PU-algebra (χ, ψ) is called a discrete dynamical system.

Note: In this study, an ordered pair (χ, ψ) means a discrete dynamical system, where χ is a PU-algebra and $\psi$is a homomorphism from χ to χ.

Definition 3.2: point $u_{0_0} \epsilon \chi$ in the discrete dynamical stucture (χ, ψ) is a fixed point if $\psi(u_{0_0}) = u_{0_0}$.

Definition 3.3: A point $u_{0_0} \in \chi$ in the discrete dynamical system (χ, ψ) is a periodic point if, for each positive integer m, ψ^m(u_{0_0}) = u_{0_0}. The period of "u_{0_0}" is defined as the lowest value of "m.".

Definition 3.4: Let (χ, ψ) represents a discrete dynamical structure then a subset "$\mathcal{A}$" of χ is an invariant subset of χ if ψ ($\mathcal{A}$)$\subset \mathcal{A}$.

Definition 3.5: Let (χ, ψ) represents a discrete dynamical structure then a subset "$\mathcal{A}$" of χ is a S-invariant of ψ if ψ ($\mathcal{A}$) = $\mathcal{A}$.

Proposition 3.1: If a mapping ψ:$\chi \longrightarrow \chi$ is a homomorphism from PU-algebra χ to χ and 0∈ χ thenψ (0) = 0.

Proof:- Let 0∈ χ then by (Definition 2.1 (P1)) ψ (0) = ψ (0∗0) as $\psi$is homomorphism therefore we haveψ (0) =ψ (0)∗ ψ(0) , so by (proposition 2.2 (C-3)) we can get ψ (0) = 0 .

Proposition 3.2: If ψ:$\chi \longrightarrow \chi$ is a homomorphism then
ψ^n:$\chi \longrightarrow \chi$ is also a homomorphism. {Where ψ^nindicates$\psi_\circ \psi_\circ \psi \dots_\circ \psi$ (n times)}.

Proof: In order to prove this, we use the principal of mathematical induction. It is required that for allu_0, v_0 in χ, $\psi^n(u_0 * v_0)$ =$\psi^n(u_0) * \psi^n(v_0)$, where n is the positive integer.
As ψ:$\chi \longrightarrow \chi$ is a homomorphism so the above statement is true at' $n = 1$ '. We presume that it is true at '$n = k$' such that
$\psi^k(u_0 * v_0) = \psi^k(u_0) * \psi^k(v_0)$ then
$\psi(\psi^k(u_0 * v_0)) = \psi(\psi^k(u_0) * \psi^k(v_0))$
$\psi^{k+1}(u_0 * v_0) = \psi(\psi^k(u_0)) * \psi(\psi^k(v_0))$ ∵ψ is homomorphism
$\psi^{k+1}(u_0 * v_0) = \psi^{k+1}(u_0) * \psi^{k+1}(v_0)$
As a result, the truth of a statement at "$n = k$" implies that it is also true at "$n = k + 1$".
Hence, if ψ is homomorphism then ψ^n is also homomorphism.

Proposition 3.3: If ψ: $\chi \longrightarrow \chi$ is a homomorphism, $\psi^n(u_0) = u_0$ let n, p be positive integers such that "n" divides "p" then $\psi^p(u_0) = u_0$.

Proof: Since $\psi^n(u_0) = u_0$.Here'n' represents a positive integer that divides 'p', then there is an integer 'q'such that$'p = nq'$ then, we have
$\psi^p(u_0) = \psi^{nq}(u_0) = \psi^{n(q-1)}(\psi^n(u_0)) = \psi^{n(q-1)}(u_0) = \psi^{n(q-2)}(\psi^n(u_0)) = \psi^{n(q-2)}(u_0) = \dots$
$= \psi^{n(q-(q-1))}(u_0) = \psi^{n(q-q+1)}(u_0) = \psi^n(u_0) = u_0$, ∵ continuously using the result $\psi^n(u_0) = u_0$.
Hence, $\psi^p(u_0) = u_0$.
Simple examples pertaining to the concepts provided in the paper are as follows:

Example (1): Let χ ={0, u_0, v_0, w_0} is a PU-algebra

Table 1: Elements of χ arranged in the following form fulfillsthe conditions of PU-algebra.

∗	0	u_0	v_0	w_0
0	0	u_0	v_0	w_0
u_0	u_0	0	w_0	v_0
v_0	v_0	w_0	0	u_0
w_0	w_0	v_0	u_0	0

and a mapping ψ: $\chi \longrightarrow \chi$ defined by

$\boldsymbol{\psi}(0) = 0, (u_0) = u_0, \boldsymbol{\psi}(v_0) = v_0$ and $\boldsymbol{\psi}\ (w_0) = w_0$
is a homomorphism. So 0, u_0, v_0, and w_0 are the enduring points as well as periodic points of period 1.

Example (2): Consider the PU-algebra $\boldsymbol{\chi}$ of example (1) then $\boldsymbol{\mathcal{A}} \subseteq \boldsymbol{\chi}$ where $\boldsymbol{\mathcal{A}} = \{0, u_0\}$ is a S-invariant of $\boldsymbol{\chi}$ as $\boldsymbol{\psi}(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{A}}$.

Theorem 3.4: Let $(\boldsymbol{\chi},\boldsymbol{\psi})$ is a discrete dynamical system then the set of all fixed points in $\boldsymbol{\chi}$ is a PU-sub-algebra of $\boldsymbol{\chi}$.

Proof: Let $\boldsymbol{\chi}$ denotes a PU-algebra and a map $\boldsymbol{\psi}$: $\boldsymbol{\chi} \longrightarrow \boldsymbol{\chi}$ is a homomorphism. Let $\boldsymbol{\chi}_f$ is a set of all enduring points in $\boldsymbol{\chi}$. We prove that $\boldsymbol{\chi}_f$ is a PU-sub-algebra for this $\boldsymbol{\chi}_f$ has to fulfill the condition of {Definition 2.3}.so we assume that $u_0, v_0 \in \boldsymbol{\chi}_f$ then $\boldsymbol{\psi}(u_0) = u_0$ and $\boldsymbol{\psi}(v_0) = v_0$ Since $\boldsymbol{\psi}$ is a homomorphism, therefore we have $\boldsymbol{\psi}(u_0 * v_0) = \boldsymbol{\psi}(u_0) * \boldsymbol{\psi}(v_0) = u_0 * v_0 \Longrightarrow u_0 * v_0$ is a fixed point $\Longrightarrow u_0 * v_0 \in \boldsymbol{\chi}_f$. Thus for any u_0, $v_0 \in \boldsymbol{\chi}_f$ we have $u_0 * v_0 \in \boldsymbol{\chi}_f$, therefore condition for sub algebra holds in $\boldsymbol{\chi}_f$. Hence $\boldsymbol{\chi}_f$ is a PU-subalgebra.

Theorem 3.5: If (χ, ψ) is a discrete dynamical system, then a PU-sub-algebra of χ is the set of all periodic points in χ.

Proof: Let the PU-algebra be $\boldsymbol{\chi}$ and a map $\boldsymbol{\psi}:\boldsymbol{\chi}\longrightarrow\boldsymbol{\chi}$ is a homomorphism. Let $\boldsymbol{\chi}_p$ be the collection of all $\boldsymbol{\chi}$ periodic points.**.** We prove that $\boldsymbol{\chi}_p$ is a PU-sub-algebra. For this $\boldsymbol{\chi}_p$ has to fulfill the condition of (Definition 2.3). So we assume that u_0, $v_0 \in \boldsymbol{\chi}_p$ and let 'u_0' and 'v_0' are having 'm' and "n" as their periods respectively. Then $\boldsymbol{\psi}^m(u_0) = u_0$ and $\boldsymbol{\psi}^n(v_0) = v_0$. Let $r = LCM[m, n]$. Now by (Proposition 4.3) the above two equations become $\boldsymbol{\psi}^r(u_0) = u_0$ and $\boldsymbol{\psi}^r(v_0) = v_0$ respectively. Next by (Proposition 4.2), we have $\boldsymbol{\psi}^r(u_0 * v_0) = \boldsymbol{\psi}^r(u_0) * \boldsymbol{\psi}^r(v_0) = u_0 * v_0 \Longrightarrow u_0 * v_0$ is a periodic point of period 'r'$\Longrightarrow u_0 * v_0 \in \boldsymbol{\chi}_p$. Thus for any u_0, $v_0 \in \boldsymbol{\chi}_p$, we have $u_0 * v_0 \in \boldsymbol{\chi}_p$, therefore condition of (Definition 2.3)holds in $\boldsymbol{\chi}_p$. Hence $\boldsymbol{\chi}_p$ is a PU-subalgebra.

Theorem 3.6: Let $(\boldsymbol{\chi}, \boldsymbol{\psi})$ be a discrete dynamical system then S-invariant subset of PU-algebra $\boldsymbol{\chi}$ is the set of all enduring points in $\boldsymbol{\chi}$.

Proof: Let $\boldsymbol{\chi}_f$ be the set of all enduring points in $\boldsymbol{\chi}$
Let $\boldsymbol{\psi}(u_0) \in \boldsymbol{\psi}(\boldsymbol{\chi}_f)$, where "$u_0$" is any element of $\boldsymbol{\chi}$
$\Longrightarrow \boldsymbol{\psi}(u_0) = u_0$, $\because \boldsymbol{\psi}(\boldsymbol{\chi}_f)$ is a collection of the images of all the enduring points
$\Longrightarrow u_0 \in \boldsymbol{\chi}_f \Longrightarrow \boldsymbol{\psi}(u_0) \in \boldsymbol{\chi}_f \quad \because \boldsymbol{\psi}(u_0) = u_0$
Thus $\boldsymbol{\psi}(u_0) \in \boldsymbol{\psi}(\boldsymbol{\chi}_f) \Longrightarrow \boldsymbol{\psi}(u_0) \in \boldsymbol{\chi}_f$. Therefore we have $\boldsymbol{\psi}(\boldsymbol{\chi}_f) \subseteq \boldsymbol{\chi}_f$ (1)
Now we suppose that $u_0 \in \boldsymbol{\chi}_f \Longrightarrow \boldsymbol{\psi}(u_0) = u_0$, where $\boldsymbol{\psi}(u_0) \in \boldsymbol{\psi}(\boldsymbol{\chi}_f) \Longrightarrow u_0 \in \boldsymbol{\psi}(\boldsymbol{\chi}_f)$.
Thus $u_0 \in \boldsymbol{\chi}_f \Longrightarrow u_0 \in \boldsymbol{\psi}(\boldsymbol{\chi}_f)$ therefore we have $\boldsymbol{\chi}_f \subseteq \boldsymbol{\psi}(\boldsymbol{\chi}_f)$. (2)
From (1) and (2) we have $\boldsymbol{\psi}(\boldsymbol{\chi}_f) = \boldsymbol{\chi}_f$. Hence $\boldsymbol{\chi}_f$ in $\boldsymbol{\chi}$ is S-invariant or strongly invariant.

Theorem 3.7: Let $(\boldsymbol{\chi},\boldsymbol{\psi})$ is a discrete dynamical structure if **"**u_0**"** is a enduring point and **"**v_0**"** is a periodic point in$\boldsymbol{\chi}$then$u_0 * v_0$is a periodic point in$\boldsymbol{\chi}$.

Proof: As "u_0" is a fixed point so we have $\boldsymbol{\psi}(u_0) = u_0$.
Assume the period of the point "v_0" is k such that $\boldsymbol{\psi}^k(v_0) = v_0$.
As "u_0" is a fixed point therefore secures a period "1" which divides "k" therefore by (proposition 4.3) we have $\boldsymbol{\psi}^k(u_0) = u_0$. Using the homomorphism characteristic of $\boldsymbol{\psi}^k$ we can get
$\boldsymbol{\psi}^k(u_0 * v_0) = \boldsymbol{\psi}^k(u_0) * \boldsymbol{\psi}^k(v_0) = u_0 * v_0 \Longrightarrow u_0 * v_0$ has a period"k". Hence $u_0 * v_0$ is a periodic point.

Theorem 3.8: Let $(\boldsymbol{\chi},\boldsymbol{\psi})$ is a discrete dynamical structure if **"**$\boldsymbol{\mathcal{A}}$**"** is a subset of $\boldsymbol{\chi}$ such that $\boldsymbol{\psi}(\boldsymbol{\chi}) \subset \boldsymbol{\mathcal{A}} \subset \boldsymbol{\chi}$, then $\boldsymbol{\mathcal{A}}$ is invariant with respect to$\boldsymbol{\chi}$.

Proof: As we have $\boldsymbol{\psi}(\boldsymbol{\chi}) \subset \boldsymbol{\mathcal{A}} \subset \boldsymbol{\chi} \Longrightarrow \boldsymbol{\mathcal{A}} \subset \boldsymbol{\chi}$ and $\boldsymbol{\psi}(\boldsymbol{\chi}) \subset \boldsymbol{\mathcal{A}}$ while $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\chi} \Longrightarrow \boldsymbol{\psi}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\psi}(\boldsymbol{\chi})$. From $\boldsymbol{\psi}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\psi}(\boldsymbol{\chi})$ and $\boldsymbol{\psi}(\boldsymbol{\chi}) \subset \boldsymbol{\mathcal{A}}$ we get $\boldsymbol{\psi}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\psi}(\boldsymbol{\chi}) \subset \boldsymbol{\mathcal{A}} \Longrightarrow \boldsymbol{\psi}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\mathcal{A}}$
Hence, $\boldsymbol{\mathcal{A}}$ is an invariant subset of PU-algebra $\boldsymbol{\chi}$.

Example 3: Consider the PU-algebra χ={ 0,u_0, v_0} and a mapping ψ: $\chi \longrightarrow \chi$ where ψ is a homomorphism given by ψ (0) = 0, ψ (u_0) = 0 *and* ψ (v_0) = 0. Let $\mathcal{A}$ = {0, v_0}, where $\mathcal{A} \subseteq \chi$ then $\psi(\chi)$ = {0} $\Longrightarrow \psi(\chi) \subset \mathcal{A} \subset \chi \Longrightarrow \mathcal{A}$ is invariant.

Theorem 3.9: Let (χ, ψ) is a discrete dynamical structure, then set of all enduring points in χ is an ideal of PU-algebra χ.

Proof: Let the PU-algebra be χ and the map ψ:$\chi \longrightarrow \chi$ be a homomorphism. Suppose that χ_f is the set of whole enduring points in χ. We prove $\chi_f$to be an ideal of χ. So χ_f has to fulfill the (Definition 2.4) conditions. As ψ is homomorphic therefore (proposition 4.1) implies that ψ(0) = 0 $\Longrightarrow$ 0 ∈ $\chi_f \Longrightarrow \chi_f \neq$ { }. Thus (I_d-1) holds in $Ϛ_f$. Now we assume that $u_0 * v_0 \in \chi_f$ and $u_0 \in \chi_f$ then we have $\psi(* \ v_0)$ = $u_0 * \ v_0$ and $\psi(u_0)$ = u_0. We know that ψ is a homomorphism i.e., $\psi(* \ v_0)$=$\psi(u_0) * \ \psi(v_0) \Longrightarrow \psi(u_0) * \ \psi(v_0)$= $u_0 * \ v_0 \Longrightarrow u_0 * \ \psi$
(v_0)= $* \ v_0$. By (Theorem 2.2 (C-2)) we get$\psi(v_0)$ = v_0 $\Longrightarrow$'v_0' is a fixed point therefore we can write ∈ χ_f. Thus u_0 $* \ v_0 \in \chi_f$ and $u_0 \in \chi_f \Longrightarrow v_0 \in \chi_f$. So ($I_d$-2) also holds in χ_f. Hence χ_f is an ideal of PU-algebra χ.

Theorem 3.10: Let (χ, ψ) is a discrete dynamical structure, then set of all periodic points in χ is an ideal of PU-algebra χ.

Proof: Let the PU-algebra be $\chi$and the map ψ:$\chi \longrightarrow \chi$ be a homomorphism.. Suppose that χ_p be the set of whole periodic points in χ. We prove that χ_p be an ideal of χ. Soχ_p has to fulfill the (definition 2.4). Since ψ is homomorphic therefore (proposition 4.1)$\Longrightarrow$ ψ(0)=0 $\Longrightarrow$ 0 secures a period "1"$\Longrightarrow$ 0 ∈ $\chi_p \Longrightarrow \chi_p \neq${}. Thus ($I_d$-1) holds in χ_p. Next assume that $u_0 *$ b∈ χ_p and $u_0 \in \chi_p$ Given corresponding periods m and n, then we have $\psi^m(u_0 * \ v_0)$ =$u_0 * \ v_0$ and $\psi^n(u_0)$ =u_0. Let **r** =LCM $[m, n]$ then according to the Proposition 4.3 the preceding equations become $\psi^r(u_0 * \ v_0)$ =$u_0 * \ v_0$ and $\psi^r(u_0)$ =u_0 respectively. By (proposition 4.2) $\psi^r(u_0 * \ v_0)$ = $u_0 * \ v_0$ becomes

$$\psi^r(u_0) * \psi^r(v_0)= u_0 * \ v_0 \qquad (3)$$

Using $\psi^r(u_0)$ =u_0 in equation (3) we get$u_0 * \psi^r(v_0)$= $u_0 * \ v_0$. By (Theorem 2.2 (C-2)) we get $\psi^r(v_0) = \ v_0 \Longrightarrow$ 'v_0' is a period of periodic point 'r' $\Longrightarrow v_0 \in \chi_p$.Thus $u_0 * v_0 \in \chi_p$ and ∈ $\chi_p \Longrightarrow \ v_0 \in \chi_p$. Hence χ_p is an ideal of PU-algebra χ.

4. CONCULUSION

This study has provided an in-depth theoretical analysis of PU-algebras, emphasizing their homomorphic properties and expanding their foundational framework by incorporating elements from discrete dynamical systems. By examining key structures such as ideals, subalgebras, periodic points, fixed points, and invariant sets, we have uncovered new insights that bridge PU-algebra theory with dynamical system concepts. These findings not only enhance the theoretical understanding of PU-algebras but also hold potential applications across fields such as artificial intelligence, computer science, and information theory.

Future research should explore the computational implications of PU-algebraic structures and their integration into algorithmic models. Further investigation into the dynamic interactions of PU-algebras could lead to innovative approaches in complex systems modeling, opening new pathways for practical applications in artificial intelligence and data-driven technologies. Through this work, we have established a basis for continued exploration, offering a promising outlook on both the theoretical and applied potentials of PU-algebra theory.

Conflicts of Interest

The authors declare no conflicts of interest.

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