Global Journal of Sciences

Global Journal of Sciences
Volume 1, Issue 2, 2024, Pages 71-79
DOI: https://doi.org/10.48165/gjs.2024.1208; ISSN: 3049-0456
https://acspublisher.com/journals/index.php/gjs

www.acspublisher.com

Research Article

Adaptation of the Optimal Auxiliary Function Method for Solving Highly Non Linear System of Fractional Order Partial Differential Equations

Muhammad Farooq[1], Ibrar Khan[1], Jamal Shah[2]
[1]*Department of Mathematics, Abdul Wali Khan University Mardan, KPK, Pakistan*
[2]*Department of Mathematics, University of Swabi, Swabi 23561, Khyber Pakhtunkhwa, Pakistan*

ARTICLE INFO

Article History
Received 19 Jul 2024
Revised 27 Sep 2024
Accepted 29 Nov 2024
Published 27 Dec 2024

Keywords
Caputo derivative
System of Fractional Order Partial Differential Equations,
Optimal Auxiliary Functions Method.

ABSTRACT

This research work represents the analysis of time-fractional system of highly non linear partial differential equation by applying the Optimal Auxiliary Function method (OAFM) while taking fractional derivative in Caputo sense. The OAFM does not require any polynomials like ADM method and small parameters like Perturbation method. The OAFM contains auxiliary constant which control the convergence of approximate solution in excellent way. Mathematica 13 is utilized to illustrate 2-dimensional curves and three-dimensional graphs and find numerical values that are displayed in a table. Furthermore, the OAFM and the exact solutions are compared. The numerical analysis reveals that the method is both valid and efficient as well as the suitability of the error bound. The results further demonstrate that the OAFM method is an effective approach for solving nonlinear physical models.

1. INTRODUCTION

The non-integer order differential equations hold a crucial role in modern times. Fractional calculus (FC) is the generalization of classical calculus. Various methods are utilized for determining the solution of fractional nonlinear partial differential equation (PDE). The homotopy analysis method (HAM) is used to explore solutions of fractional partial differential equations [1]. The radial basis functions method is considered a powerful tool for scattered data interpolation problems. Radial basis functions are applied as a meshless method for obtaining numerical solutions to partial differential equations, especially using a collocation approach. Due to the collocation method, this approach eliminates the need to compute any integrals. The meshless property of these methods is the basic advantage of the numerical approaches which utilize radial basis functions over traditional approaches. Radial basis functions (RBFs) are employed for solving partial differential equations [2, 3]. Suleman et. al [4] have been utilized a new projected differential transform method for space and time fractional telegraph equations. In [5] an effective method separation of variables has been adopted to calculate approximate solutions of TFT equations with different kinds of boundary conditions.

Similarly, utilizing fractional integral in the sense of Riemann–Liouville's (R-L) and the Caputo derivative, we investigate highly nonlinear system of partial differential equations by applying a significant technique known as optimal auxiliary function method (OAFM). The OAFM was initially introduced by Vasile Marinca and co-authors for the study of thin film flow problem [6].

Later on, the OAFM have gained the attention of many researchers. For example, in ref. [7] the OAFM is adopted for the approximate solution of generalized seventh order Korteweg-Devries equation arising in shallow water waves. The numerical solution of FPDEs has been studied in [8] by applying the Bernoulli wavelets and the collocation approach. Y. Khan at.al [9] implemented an analytical approach named Auxiliary Laplace parameter method for solving fractional differential equations (FDEs). In [10], the simple equation method is analyzed for obtaining the approximate solution of the Kodomtsev–Petviashvili (KP) equation. A modified variational iteration method [11] has been investigated for finding the solution of nonlinear partial differential equation. The iterative Laplace transform approach is studied for solution of linear

*Corresponding author. Email: ibrar.khan@awkum.edu.pk

and nonlinear in [12]. For the problems arising from physics, the ADM has been adopted to study the dynamics of waves on shallow water surfaces [13].

The Adomian decomposition method (ADM) entails decomposing the problem under examination into linear and nonlinear components. This approach produces a solution represented as a series, with its terms defined by a recursive relation utilizing the Adomian polynomials. Andrianov [14] presents essential studies on many facets of the variations of the ADM. The Laplace residual power series method (LRPSM) is a powerful approach which is independent from the use of any type of polynomials or small parameters dealing with nonlinear term. The LRPSM can handle both linear and nonlinear and nonlinear differential equations of integer as well as fractional order. LRPSM is based on the residual power series with the aid of Laplace transform which provides an effective procedure dealing with highlynonlinear physical models. [15-17]. The new iterative method (NIM) is a newly developed technique for mathematical analysis of non-integer order partial differential equations. Basically, NIM is an extension of the ADM technique which does not need any type of polynomial like ADM approach or discretization as needed for numerical approaches. NIM technique overcomes the computation process and provides an accurate solution for a wide range of complicated mathematical and physical phenomena [18,19].

The OAFM was applied to find solutions of different types of differential equations including ordinary and partial differential equation of integer order and fractional order in Physics and engineering. As compared to other techniques, the OAFM does not require any assumption and small parameters. This method involves auxiliary constant which controls the convergence of approximate solutions in a precise and controlled way. Several researchers have been focused recently on the OAFM approach. The analysis reveals that the OAFM is a full power approach for solving linear and nonlinear differential equations.

For instance, the Blasius problem's nonlinear differential equation is solved using the OAFM. In Ref. [20], optimum auxiliary functions are used to simulate the thin film flow of a third-grade fluid on a moving belt issue. The optimum auxiliary functions technique is used in Ref. [21] to provide an analytic approximation solution to the boundary nonlinear issue of the viscous flow caused by a stretched surface with partial slip Ref. [22].

The main goal of this paper is to find the approximate solution of system of highly non linear partial differential equation using OAFM technique. Furthermore, for the analysis of the OAFM we have made a comparison of the OAFM and exact solution using different tables, 2D and 3D graphs.

Novelty

- **Pioneering Application of OAFM to** Whitham–Broer–Kaup **Equations**: In this manuscript for the first time the Optimal Auxiliary Functions Method (OAFM) is utilized for solving the approximate solution of nonlinear Whitham–Broer–Kaup Equations of fractional order, marking a significant advancement in the tools and techniques for studying these equations.
- **Expanded Research Base**: The Optimal Auxiliary Functions Method has been successfully applied in different filed of science and engineering, including Partial Differential Equations (PDEs) and the system of partial differential equation. These applications underscore the method's adaptability and efficacy in handling both linear and nonlinear dynamical systems, further solidifying its capability as a powerful analytical tool.

The remaining paper is arranged in the following manner: Section 2 discusses the basic definition related to fractional Calculus. In Section 3, the general procedure of OAFM is briefly reviewed, and in Section 4, the proposed equations are examined by addressing the approximations to the solutions of the nonlinear Whitham–Broer–Kaup **equation**. Section 5 also includes results and discussions. Finally, Section 6 discusses the conclusion.

2. BASIC PRELIMINARIES

Definition 2.1. The Reiman–Liouville fractional integral operator is given as:

$$J_t^{\mu} u(\eta,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-r)^{\mu-1}\, u(\eta,r)dr. \tag{1}$$

Definition 2.2. Using the Caputo formula, the fractional derivative of $u(\eta,t)$ is given by

$$c_{D_t^{\propto}} u(\eta,t) = \frac{1}{\Gamma(w-\mu)} \int_0^t (t-r)^{w-\mu-1}\, v(\eta,r)dr, \qquad w-1<\mu\leq w, t>0. \tag{2}$$

Lemma1. For$w-1<\propto\leq w, \beta>-1, t\geq -1, and\ \lambda \epsilon R,$ we have

1. $D_t^{\propto} t^{\beta} = \dfrac{\Gamma(\propto+1)}{\Gamma(\beta-\propto+1)} t^{\beta-\propto}.$
2. $D_t^{\propto}\lambda = 0.$

3. $D_t^{\propto} I_t^{\propto} u(\eta, t) = u(\eta, t)$.
4. $I_t^{\propto} D_t^{\propto} u(\eta, t) = u(\eta, t) - \sum_{i=0}^{n-1} \partial^i u(\eta, 0) \frac{t^i}{i!}$.

3. BASIC IDEA OF OAFM

In this part, we explain the OAFM procedure for solving differential equations of fractional order.

$$\frac{D^{\mu} u(\zeta,t)}{\partial t^{\mu}} = X(\zeta, t) + N(u(\zeta, t)). \tag{3}$$

With the initial condition:

$$u(\zeta, 0) = w(\zeta) \tag{4}$$

Here, $\frac{\partial^{\mu}}{\partial t^{\mu}}$ represents the Caputo operator, $u\ (\zeta, t)$ is unknown function and $N(u(\zeta, t))$ denotes the non linear term.

Step 1: For solving equation (3), we will use an approximate solution that consists of two components.

$$\tilde{u}(\zeta, t) = u_0(\zeta, t) + u_1(\zeta, t, c_i), \quad i = 1,2,3, \dots, q, \tag{5}$$

Step 2: For the determination of zero and the first-order solutions, we substitute Eq.(5) into eq.(3).

$$\frac{\partial^{\mu} u_0(\zeta,t)}{\partial t^{\mu}} + \frac{\partial^{\mu} u_1(\zeta,t)}{\partial t^{\mu}} + X(\zeta, t) + N\left[\left(\frac{\partial^{\mu} u_1(\zeta,t)}{\partial t^{\mu}} + \frac{\partial^{\mu} u_1(\zeta,t,c_i)}{\partial t^{\mu}}\right)\right] = 0 \tag{6}$$

For the purpose of finding the zeroth order approximation $u_0(\zeta, t)$, we will solve the linear equation

$$\frac{\partial^{\mu} u_0(\zeta,t)}{\partial t^{\mu}} + X(\zeta, t) = 0 \tag{7}$$

Applying inverse operator on Eq. (7),

$$u_0(\zeta, t) = X(\zeta, t). \tag{8}$$

Step 3: The nonlinear term present in the extending form in eq. (2.4) are,

$$N \frac{\partial^{\mu} u_1(\zeta,t)}{\partial \beta^{\mu}} + \frac{\partial^{\mu} u_1(\zeta,t)}{\partial \beta^{\mu}} = N[u_0(\zeta, t)] + \sum_{k=1}^{\infty} \frac{u_1^k}{k!} N^{(k)}[u_0(\zeta, \beta)] \tag{9}$$

Step 4: We propose an alternative equation to solve (9) more accurately and achieve quickly convergence for the first order approximation $u_1(\zeta, t)$. This equation can be written as follows.

$$\frac{\partial^{\mu} u_1(\zeta,t,c_i)}{\partial t^{\mu}} = -\Delta_1[u_0(\zeta, t)] N[u_0(\zeta, t)] - \Delta_2[u_0(\zeta, t), c_j], \tag{10}$$

Step 5: We obtain the first-order solution $u_1(\zeta, t, c_i)$ by using the inverse operator after substituting the auxiliary function, along with the auxiliary constant, into Eq. (10).

Step 6: Different approaches, including least-squares, Galerkin's, Ritz, and collocation are employed to find the numerical values of convergence control parameters c_i. Among these, the least-squares method proves to be the most effective in minimizing errors.

$$G(c_i, c_j) = \int_0^t \int_{\omega} R^2(\zeta, t, C_i, C_j) d\zeta dt \tag{11}$$

Where R represents the residual,

$$R(\zeta, t, c_i, c_j) = \frac{\partial u(\zeta, t, c_i, c_j)}{\partial t} + X(\zeta, t) + N[u(\zeta, t, c_i, c_j)],$$
$$i = 1,2, \dots, , j = y+1, y+2, \dots, r \tag{12}$$

4. APPLICATION OF OPTIMAL AUXILIARY FUNCTION METHOD

To demonstrate the effectiveness and accuracy of the proposed method, we offer approximate solutions for highly nonlinear system of partial differential equation in this section. Mathematica 13 is used to perform all computations.

Example 1: Consider fractional order system of nonlinear partial differential equation [23].

$$D_t^{\mu} u + u\partial_{\zeta} u + \partial_{\zeta} u + \partial_{\zeta} v = 0. \tag{13}$$

$$D_t{}^{\mu} v + u\partial_{\zeta} v + v\partial_{\zeta} u - \partial_{\zeta,\zeta} v + 3\partial_{\zeta,\zeta,\zeta} u = 0. \tag{14}$$

With the initial conditions:

$$u(\zeta, 0) = \frac{1}{2} - 8\tanh[-2\zeta], \tag{15}$$

$$v(\zeta, 0) = 16 - 16\tanh[-2\zeta]^2. \tag{16}$$

From equations (13) and (14), the linear terms are:

$$L(u(\zeta, t)) = D_t^{\mu} u(\zeta, t), \tag{17}$$

$$L(v(\zeta, t)) = D_t^{\mu} v(\zeta, t). \tag{18}$$

We consider the nonlinear terms from Eq.(13) and (14), we obtain

$$N(u(\zeta,t)) = u\partial_\zeta u + \partial_\zeta u + \partial_\zeta v, \tag{15}$$
$$N(v(\zeta,t)) = u\partial_\zeta v + v\partial_\zeta u - \partial_{\zeta,\zeta} v + 3\partial_{\zeta,\zeta,\zeta} u\,. \tag{16}$$

The zeroth order problem.

$$D_t^{\mu}\, u_0(\zeta,t) = 0, \tag{17}$$
$$D_t^{\mu} v_0(\zeta,t) = 0\,. \tag{18}$$

By utilizing the inverse operator, we obtain the subsequent zeroth order approximation solution:

$$u_0(\zeta,t) = \frac{1}{2} - 8\tanh[-2\zeta], \tag{19}$$
$$v_0(\zeta,t) = 16 - 16\tanh[-2\zeta]^2. \tag{20}$$

By substituting equations (19) and (20) into equations (17) and (18), the equation for the nonlinear terms is obtained.

$$N(u_0(\zeta,\mathrm{t})) = 16\mathrm{Sech}[2\zeta]^2 - 64\mathrm{Sech}[2\zeta]^2\mathrm{Tanh}[2\zeta] + 16\mathrm{Sech}[2\zeta]^2(\frac{1}{2} + 8\mathrm{Tanh}[2\zeta]),$$
$$N(v_0(\zeta,\mathrm{t})) = 128Sech[2\zeta]^4 - 256Sech[2\zeta]^2Tanh[2\zeta]^2 + 16Sech[2\zeta]^2(16 - 16Tanh[2\zeta]^2)$$
$$+3(-128\mathrm{Sech}[2\zeta]^4 + 256\mathrm{Sech}[2\zeta]^2\mathrm{Tanh}[2\zeta]^2). \tag{21}$$

The auxiliary function A_1 and A_2 are selected as:

$$A_1 = c_1\left(\frac{1}{2} - 8\tanh[-2\zeta]\right) + c_2\left(\frac{1}{2} - 8\tanh[-2\zeta]\right)^3,$$
$$A_2 = c_3\left(\frac{1}{2} - 8\tanh[-2\zeta]\right)^5, \qquad A_3 = c_4(16 - 16\tanh[-2\zeta]^2)^6 + c_5(16 - 16\tanh[-2\zeta]^2)^7,$$
$$A_4 = c_6(16 - 16\tanh[-2\zeta]^2)^9\,. \tag{22}$$

The first-order approximation is determined using the OAFM approach as detailed in the corresponding part.

$$(u_1)^{(0,\mu)}[\zeta,t] = -(A_1\mathrm{N}[u_0(\zeta,t)] + A_2),$$
$$(v_1)^{(0,\mu)}[\zeta,t] = -(A_3\mathrm{N}[v_0(\zeta,t)] + A_4).$$

$$u_1(\zeta,t) = \Bigg(-\frac{1}{32\mu\Gamma[\mu]}\big(t^{\mu}(1 + 16\mathrm{Tanh}[2\zeta])(16\mathrm{Sech}[2\zeta]^4(4\mathrm{c1} - 255\mathrm{c2} + (4\mathrm{c1} + 257\mathrm{c2})\mathrm{Cosh}[4\zeta] + 32\mathrm{c2Sinh}[4\zeta])(3 + 8\mathrm{Tanh}[2\zeta]) + \mathrm{c3}(1 + 16\mathrm{Tanh}[2\zeta])^4)\big)\Bigg). \tag{23}$$

$$v_1(\zeta,t) = -\frac{1}{\mu\Gamma[\mu]}(1073741824t^{\mu}(-\mathrm{c4} - 64\mathrm{c5} + 2\mathrm{c6} + 2(32\mathrm{c5} + \mathrm{c6})\mathrm{Cosh}[4\zeta]. + \mathrm{c4Cosh}[8\zeta])\mathrm{Sech}[2\zeta]^{18}). \tag{24}$$

According to the OAFM procedure

$$u(\zeta,t) = u_0(\zeta,t) + u_1(\zeta,t) \tag{25}$$
$$v(\zeta,t) = v_0(\zeta,t) + v_1\,(\zeta,t) \tag{26}$$

By utilizing equations (4.11,4.12) and (4.16,4.17), we obtain the solution

$$u(\zeta,t) = \Big(\frac{1}{2} + 8\tanh[2\zeta] - \frac{1}{32\mu\Gamma[\mu]}\big(t^{\mu}(1 + 16\mathrm{Tanh}[2\zeta])(16\mathrm{Sech}[2\zeta]^4(4\mathrm{c1} - 255\mathrm{c2} + (4\mathrm{c1} + 257\mathrm{c2})\mathrm{Cosh}[4\zeta] + 32\mathrm{c2Sinh}[4\zeta])(3 + 8\mathrm{Tanh}[2\zeta]) + \mathrm{c3}(1 + 16\mathrm{Tanh}[2\zeta])^4)\big)\Big), \tag{27}$$

$$v(\zeta,t) = \begin{pmatrix} 16 - 16\tanh[-2\zeta]^2 - \frac{1}{\mu\Gamma[\mu]}(1073741824t^{\mu}(-\mathrm{c4} - 64\mathrm{c5} + 2\mathrm{c6} \\ +2(32\mathrm{c5} + \mathrm{c6})\mathrm{Cosh}[4\zeta] \end{pmatrix} \tag{28}$$

The exact solution is

$$u(\zeta,t) = \frac{1}{2} - 8\tanh\left\{-2\left(\zeta - \frac{t}{2}\right)\right\}, \tag{29}$$
$$v(\zeta,t) = 16 - 16\tanh^2\left\{-2\left(\zeta - \frac{t}{2}\right)\right\}. \tag{30}$$

Table 1: Numerical observation using OAFM and exact solution $t = 0.0003$ and $\mu = 1$.

ζ	OAFM $u(\zeta,t)$	Exact $u(\zeta,t)$	Absolute error
0.	0.4996095370832	0.497600000072	0.002009537011
0.1	2.0765487958072	2.076695922127	0.000147126319
0.2	3.53439326574132	3.53753793091881	0.003144665177
0.3	4.78880616982925	4.7946884742111	0.005882304381
0.4	5.80347994365207	5.810952162435	0.007472218787
0.5	6.58403602267938	6.5917450789118	0.0077090562324
0.6	7.16155040684451	7.1685046249999	0.006954218155
0.7	7.57656116652452	7.58229428190833	0.005733115383
0.8	7.86853490067131	7.872987070363	0.004452169691
0.9	8.07087083346198	8.074199492064	0.0033286586
1.	8.20961552878327	8.2120510295790	0.00243550079

Table 2: Numerical observation using OAFM solution $t = 0.0003$ and $\mu = 0.5$, 0.7 and 1.

ζ	$u(\zeta,t)$ OAFM(μ =0.5)	$u(\zeta,t)$ OAFM(μ =0.7)	$u(\zeta,t)$ OAFM(μ =1.0)
0.1	0.7737878945	1.496843530021	1.91541816233115
0.2	0.7744257827	2.3062571728076	3.19302954467703
0.3	0.75890589202	2.9955727304476	4.29037212564937
0.4	1.1238082626	3.7211098686454	5.22467959609516
0.5	1.955860438309	4.524580788280	6.01160491653026
0.6	3.08063739866	5.345617216192	6.6568069059150
0.7	4.25722017987	6.099513884050	7.16601191271277
0.8	5.31291848549	6.731331372542	7.55244612988835
0.9	6.17161613556	7.2257365126598	7.83596348222616
1.	6.82650163864	7.5941546222345	8.038546519055533

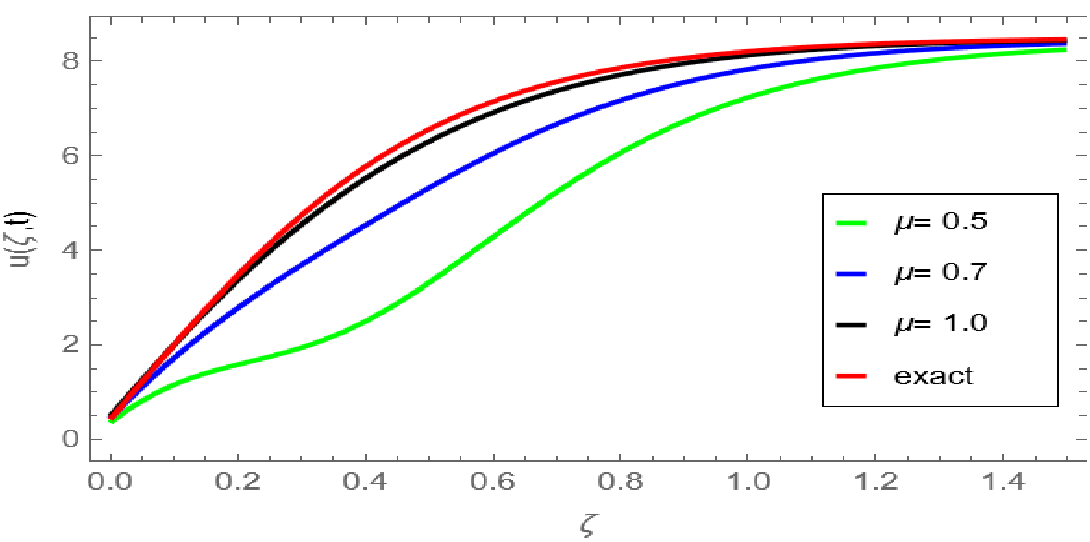

Figure 1. Variation in $u(\zeta,t)$ approximate solution using OAFM technique.

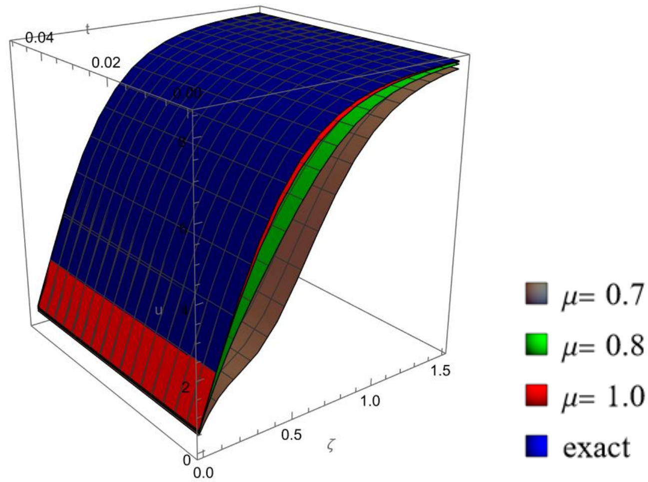

Figure 2: 3D Variation in $u(\zeta,t)$ approximate solution using OAFM technique.

Table 3: Numerical observation for $v(\zeta,t)$ using OAFM and exact solution $t = 0.002$ and $\mu = 1$.

ζ	OAFM $v(\zeta,t)$	Exact $v(\zeta,t)$	Absolute error
1.	1.13041318578846	1.134780289744	0.00436710395629
1.1	0.76680550630993	0.7698040282552	0.0029985219453632
1.2	0.51814038940530	0.5201830648402	0.0020426754349340
1.3	0.34919676312190	0.350580929361	0.0013841662397275
1.4	0.23492264165093	0.2358572484965	0.0009346068455720
1.5	0.15785659464702	0.1584861442578	0.00062954961080663
1.6	0.10598716538568	0.106410548132	0.00042338274637829
1.7	0.07112309097189	0.071407516258	0.00028442528633121
1.8	0.04771022558820	0.047901162223	0.00019093663542868
1.9	0.03199685797964	0.032124972918	0.00012811493927422
2.	0.02145521092841	0.021541145736	0.0000859348081050

Table 4: Numerical observation using OAFM and exact solution $t = 0.2$ and $\mu = 0.6, 0.8$ and 1.0

ζ	$v(\zeta,t)$ OAFM(μ =0.5)	$v(\zeta,t)$ OAFM(μ =0.7)	$v(\zeta,t)$ OAFM(μ =1.0)
0.	−20.91016069251817	−9.66405552951524	−1.324469775051064
0.1	−6.189086224563302	0.381759234669318	5.254390994051676
0.2	8.994653348633198	10.42533915066570	11.486268846841385
0.3	10.111162289797267	10.49936052289297	10.787230181562315
0.4	8.291412143193797	8.49051716689871	8.63816414291973
0.5	6.528866870022172	6.586977870600604	6.630070271532071
0.6	4.847043072381775	4.857182154359329	4.8647008234805735
0.7	3.454336553359001	3.455586627017306	3.4565136231925813
0.8	2.408032162514898	2.40815435805367	2.4082449725517314
0.9	1.6569004369254632	1.656910658522607	1.6569182383811185
1.	1.1304106703898533	1.13041144041739	1.1304120114338243

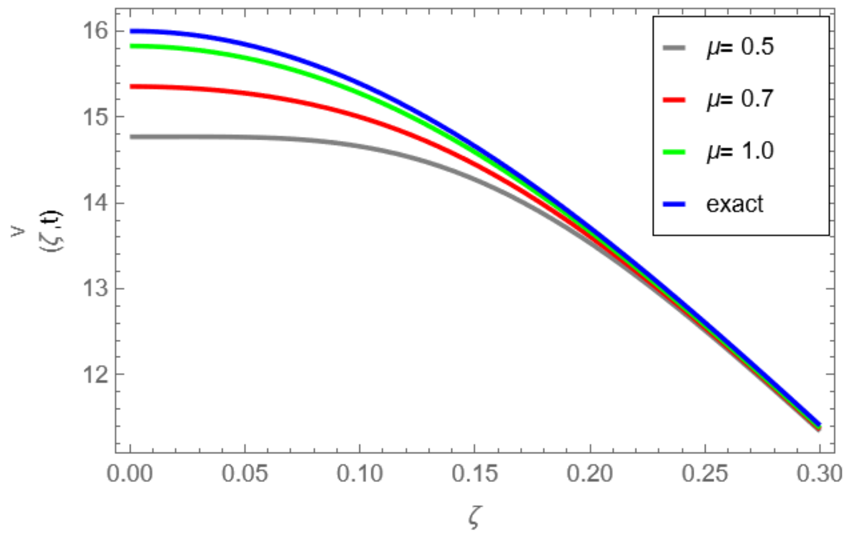

Figure 3. Variation in $v(\zeta,t)$ approximate solution using OAFM technique and exact solution.

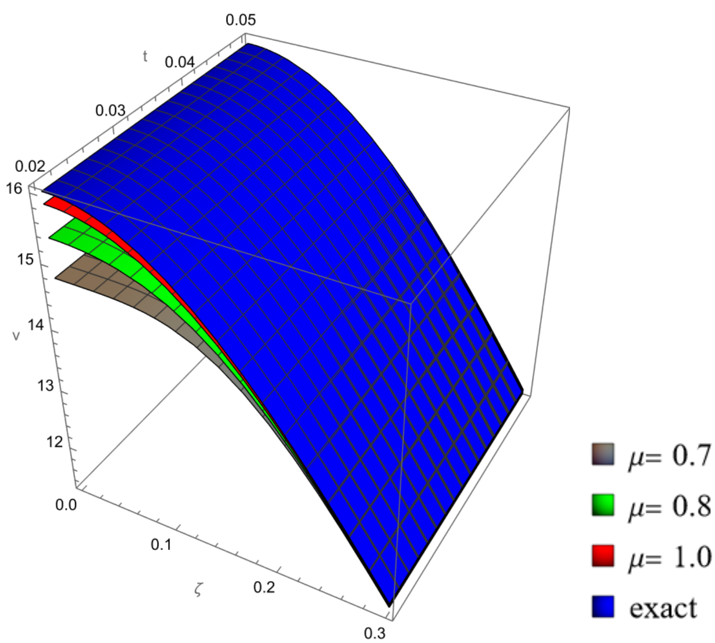

Figure 4. Variation in $v(\zeta,t)$ approximate solution using OAFM technique and exact solution

5. DISCUSSION AND RESULTS

We have successfully determined the approximate solution of non linear system of partial differential equation of non integer order by applying the OAFM approach.

The numerical values of auxiliary constants for $u(\zeta,t)$ are obtained using collocation method:

$c1 = 0.1082604514927619$,
$c2 = 0.0008057121984888594$,
$c3 = 0.000016037682300631862$,
$c4 = -8.450462288846204 \times 10^{-7}$,
$c5 = 5.054632913520107 \times 10^{-8}$
$c6 = -1.376064359519054 \times 10^{-7}$.

In Figure 1, the 2D comparison of exact and our obtained solution was made by varying the value of fractional parameter $\mu = 0.5, \mu = 0.7$ and $\mu = 1.0$. In Figure 2, the 3D comparison of exact and our obtained solution was made by varying the value of fractional parameter $\mu = 0.7, \mu = 0.8$ and $\mu = 1.0$. Furthermore Table (1) showcases the solution obtained by OAFM, exact and absolute error at integer order $\mu = 1$. Table (2) showcases the solution obtained by OAFM, at $\mu = 0.5$, 0.7 and 1.

The numerical values of auxiliary constants for $v(\zeta, t)$ are :

$c1 = -0.00671381932359124$,

$c2 = -0.0000528119421415489 5$,

$c3 = -0.0000942093107053203 6$,

$c4 = 9.432802248115106 \times 10^{-8}$,

$c5 = -9.322783371733257 \times 10^{-9}$,

$c6 = 2.016833724343574 \times 10^{-8}$.

In Figure 3, the 2D comparison of exact and our obtained solution was illustrated by varying the value of fractional parameter $\mu = 0.5, \mu = 0.7$ and $\mu = 1.0$. In Figure 4, the 3D comparison of exact and our obtained solution was made by varying the value of fractional parameter $\mu = 0.7, \mu = 0.8$ and $\mu = 1.0$. Table (13) presents the solution obtained by OAFM, exact and absolute error at fractional parameter $\mu = 1$. Table (4) gives the solution obtained by OAFM, at $\mu = 0.5$, 0.7 and 1. These 2D and 3D graph visualize that the OAFM solution converges to exact solution as the value of fractional parameter close to integer order.

6. CONCLUSION

We have applied Optimal Auxiliary Function method (OAFM) for solving the time-fractional order system of partial differential equation. The approximate solutions are obtained while taking fractional derivative in Caputo sense. The OAFM does not require any polynomials like other methods and small parameters. Due to auxiliary constant which controls the convergence of approximate solution in excellent way. Mathematica 13 is used to illustrate 2-dimensional curves and three-dimensional graphs and find numerical values that are displayed in a table. Furthermore, the OAFM method and the exact solutions are compared. The numerical analysis reveals that the method is both valid and efficient as well as the suitability of the error bound. The results further demonstrate that the OAFM method is an effective approach for solving non linear fractional order partial differential equations.

Conflicts of Interest

The authors declare no conflicts of interest.

Funding

The authors did not receive financial assistance from any public, private, or not-for-profit funding bodies for this research.

Acknowledgment

All authors have contributed equally.

References

[1] Cahn, J. W., & Hilliard, J. E. (1958). Free energy of a nonuniform system. I. Interfacial free energy. The Journal of chemical physics, 28(2), 258-267..

[2] Kansa, E. J. (1990). Multiquadrics—A scattered data approximation scheme with applications to computational fluid-dynamics—II solutions to parabolic, hyperbolic and elliptic partial differential equations. Computers & mathematics with applications, 19(8-9), 147-161

[3] Zerroukat M, Power H, Chen C. A numerical method for heat transfer problems using collocation and radial basis functions. International Journal for numerical methods in Engineering. 1998 Aug 15;42(7):1263-78.

[4] Suleman, M., Elzaki, T. M., Rahman, J. U., & Wu, Q. (2016). A novel technique to solve space and time fractional telegraph equation. Journal of Computational and Theoretical Nanoscience, 13(3), 1536-1545.

[5] Chen, J., Liu, F., & Anh, V. (2008). Analytical solution for the time-fractional telegraph equation by the method of separating variables. Journal of Mathematical Analysis and Applications, 338(2), 1364-1377.

[6] Marinca, B., & Marinca, V. (2018). Approximate analytical solutions for thin film flow of a fourth grade fluid down a vertical cylinder. Proceed Romanian Academy, Series A, 19(1), 69-76.

[7] Zada, L., Nawaz, R., Ayaz, M., Ahmad, H., Alrabaiah, H., & Chu, Y. M. (2021). New algorithm for the approximate solution of generalized seventh order Korteweg-Devries equation arising in shallow water waves. Results in Physics, 20, 103744.

[8] Mohammed, O. H., Hanan, I. K., & Al-Sabbagh, A. A. Some Numerical Methods For Solving Fractional Parabolic Partial Differential Equations.

[9] Khan, Y., Vazquez-Leal, H., & Faraz, N. (2013). An auxiliary parameter method using Adomian polynomials and Laplace transformation for nonlinear differential equations. Applied Mathematical Modelling, 37(5), 2702-2708.

[10] Nofal, T. A. (2016). Simple equation method for nonlinear partial differential equations and its applications. Journal of the Egyptian Mathematical Society, 24(2), 204-209..

[11] Abassy, T. A., El-Tawil, M. A., & El-Zoheiry, H. (2007). Modified variational iteration method for Boussinesq equation. Computers & Mathematics with Applications, 54(7-8), 955-965.

[12] Jafari, H., Nazari, M., Baleanu, D., & Khalique, C. M. (2013). A new approach for solving a system of fractional partial differential equations. Computers & Mathematics with Applications, 66(5), 838-843..

[13] Syam, M. I. (2005). Adomian decomposition method for approximating the solution of the Korteweg–deVries equation. Applied Mathematics and Computation, 162(3), 1465-1473.

[14] Andrianov, I. V., Olevskii, V. I., & Tokarzewski, S. (1998). A modified Adomian's decomposition method. Journal of applied mathematics and mechanics, 62(2), 309-314..

[15] Shafee, A., Alkhezi, Y., & Shah, R. (2023). Efficient solution of fractional system partial differential equations using Laplace residual power series method. Fractal and Fractional, 7(6), 429.

[16] Alkhezi, Y., Shafee, A., & Shah, R. (2022). Fractional view analysis of partial differential equation via residual power series transform method. Appl Math Sci, 16(12), 585-99.

[17] Alquran, M., Ali, M., Alsukhour, M., & Jaradat, I. (2020). Promoted residual power series technique with Laplace transform to solve some time-fractional problems arising in physics. Results in Physics, 19, 103667.

[18] Falade, K. I., & Tiamiyu, A. T. (2020). Numerical solution of partial differential equations with fractional variable coefficients using new iterative method (NIM). Mathematical Sciences and Computing, 3, 12-21..

[19] Al-Jawary, M. A. (2016). An efficient iterative method for solving the Fokker–Planck equation. Results in physics, 6, 985-991.

[20] Marinca, V., & Herisanu, N. (2015). An application of the optimal auxiliary functions to Blasius problem. The Romanian Journal of Technical Sciences. Applied Mechanics., 60(3), 206-215..

[21] I Marinca, V., & Marinca, B. (2018). Optimal auxiliary functions method for nonlinear thin film flow of a third grade fluid on a moving belt. A A, 2(2), 1-2..

[22] Marinca, V., Ene, R. D., & Marinca, V. B. (2018). Optimal Auxiliary Functions Method for viscous flow due to a stretching surface with partial slip. Open Engineering, 8(1), 261-274..

[23] Nonlaopon, K., Naeem, M., Zidan, A. M., Shah, R., Alsanad, A., & Gumaei, A. (2021). Numerical Investigation of the Time-Fractional Whitham–Broer–Kaup Equation Involving without Singular Kernel Operators. Complexity, 2021(1), 7979365.