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Research Article

# On Double Delay Terms Robbin Boundary Value Problem with Fractional Derivative

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**ABSTRACT**

In this manuscript, the existence theory of a mixed delay Robbin boundary value problem (RBVP) together with impulsive conditions is investigated. For the required results, we use topological degree of non-compactness. The concerned tools have the ability to relax the strong compact criteria to some weaker one. As compared to fixed point theory, the aforesaid proposed tools are more powerful and applicable in dealing various nonlinear problems of fractional order differential equations (FODEs) with different kinds of boundary conditions. Proper example is given to illustrate the results.

## 1. INTRODUCTION

One of the most practical areas of differential equations in research is BVPS. The aforesaid area has been used very well to study various real world problems. Also various problems of physics, fluid mechanics and engineering disciplines have been investigated by using the concept of BVPs. For some details, we refer here few references like applications of fractional calculus in various fields in [1], applications in science and engineering [2], uses of the mentioned area in physics [3], some diverse problems and detail utilizations, theory and importance of fractional order derivatives and integrals in [4].

Recently the area of fractional calculus has given very high dedications from researchers. This is due to the important applications of the mentioned area in modeling various real-world process and phenomenons. Here for the references, we give some remarkable contribution like [5-7]. Keeping in mind the importance of the mentioned area, researchers have studied FODEs from various aspects. The areas which have been very explored are the qualitative analysis, stability theory, numerical and analytical approximations of FODEs. For the aforesaid studies, authors have used various tools of nonlinear functional analysis and methods of numerical sides. Here, we refer some work as [8-12]. An important area of differential equations which increasingly used to model those evolution processes which sufferer from abrupt changes or they behave variously is known as impulsive equations. The said area have crossover behaviors in nature. In real world applications like sudden change in season, earthquake, fluctuation in economy of less developed countries, heart beat, pendulum motions, etc are the important examples. Now to model such situation with classical differential equations is not adequate and hence we cannot obtain well informative results about the phenomenon. Therefore, it has been found

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that modeling the aforesaid process by using impulsive equations will give accurate and well informative results. It is the ability of impulsive equations which model the evolutions processes with crossover behaviors in nature more brilliantly. Hence, researchers have extended the mentioned area to FODEs. Here, the use of fractional order globalize the nature of the operator from local to nonlocal. Also, fractional order derivatives and integrals provide a complete spectrum of the function on whom these operators work. In this regards, some important work been done like [13-15]. The area devoted to establish the existence theory by using fixed point theory has been explored very well. Plenty of research work has been published in this regards. In the same line, the mentioned theory have also been utilized to study various BVPs of impulsive FODEs. Here we refer some work like [16-18]. On the other hand delay differential equations constitutes an important class which model those process which involve delay concept. Various delay concept like proportional and discrete type delay problems have increasingly investigated in literature. Here we refer some work on delay FODEs as [19-21].

In the same way, BVPs with both kinds of delay concepts are very rarely studied. In this regards some problems under impulsive criteria have been recently considered like [22-23]. Also, it is interesting to mentioned that BVPs with impulsive conditions of FODEs have increasingly considered in literature for the existence theory. Robbin BVPs have important applications in mathematical physics and engineering disciplines. The concerned area has very rarely considered together with impulsive conditions. Here, we mentioned that the degree theory have been used very well in various articles. For instance [24] author used it to deal a nonlinear integral system. Further, authors [25-27] have used it to investigate different classes of FODEs. In the same way, authors [28] used it to study an impulsive problem of FODEs.

Recently, authors [29] have considered a class of Robbin BVPs under impulsive FODEs. They apply topological degree theory to establish the existence results for the given problem. Inspired from the applicability and uses of FODEs and degree theory, here we extend the problem been studied in [29] under Robbin boundary conditions together with impulsive behavior involving mixed delays terms as

$$\begin{cases} D^{\alpha}\aleph(t) = \Xi(t, \aleph(t), \aleph(\lambda t), \aleph(\tau - t)), , t \neq t_i, \ 1 < \alpha \leq 2, \\ \triangle\aleph(t_i) = I_i(\aleph(t_j)), \triangle\aleph^{'}(t_i) = J_j(\aleph(t_i)), \\ \triangle_1\aleph(0) + \triangle_2\aleph(t) = g_1(\aleph), \triangle_3\aleph^{'}(0) + \triangle_4\aleph^{'}(t) = g_2(\aleph), \end{cases} \tag{1}$$

where $j = 1, 2, \ldots, l,\ 0 < \alpha \leq 2$ as well as the nonlinear mapping $f : [0, \tau]\times R^3 \to R$ is continuous and $I_j, J_j$ are a nonlinear map that establish the magnitude at the discontinuity of $t_i$, where $0 < t_1 < t_0 < t_2 < t_3 \ldots, < t_l$ and $I_i(\aleph(t_i)) = \aleph(t_i^+) - \aleph(t_i^-)$, $J_j(\aleph^{'}(t_i)) = \aleph^{'}(t_i^+) - \aleph^{'}(t_i^-)$, the symbols $\aleph(t_i^+), \aleph^-(t_i^+)$, and $\aleph(t_i^-), \aleph^-(t_i^-)$ are the limits from the right and left, respectively, and $D^{\alpha}$ represent Caputo derivative of Different order where $1 < \alpha \leq 2$. We prove the existence and uniqueness of solution to the proposed problems by the mentioned degree theory. We provide an example also.

## 2. PRELIMINARIES

Here we recollect some already defined results from (4), (5), (7).

**Definition 1.** *Let $\aleph \in L([0, t], R)$ be a function. Then Riemann-Liouville integral of fractional order $\alpha > 0$ of function $\aleph$ is given by*

$$I^{\alpha}\aleph(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1}\aleph(\zeta)\, d\zeta$$

*provided that integral is pointwise defined on right side.*

**Definition 2.** *The Caputo derivative of fractional order $\alpha > 0$ of function $\aleph \in C[0, t]$ is defined by*

$$D^{\alpha}\aleph(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \zeta)^{n-\alpha-1}\aleph^{(n)}(\zeta) d\zeta,$$

*provided that the integral on right hand side is pointwise defined on $(0, \infty)$, where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number $\alpha$.*

**Lemma 1.** *Let $\alpha > 0$, then FODE*

$$D^{\alpha}\aleph(t) = 0$$

*has the solution in the form of*

$$\aleph(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1}.$$

**Lemma 2.** *suppose $\alpha > 0$, then*

$$D^{\alpha}[\aleph(t)] = h(t)$$

*has a solution*

$$\aleph(t) = I^{\alpha}[h(t)] + [a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1}].$$

## 3. THE EXISTENCE OF SOLUTION TO THE GIVEN PROBLEM

In this section,we introduce several hypothesis for the existance and uniqueness theories of the problems being studied and provide space $PC(J, R) = \Psi$ , a name that we required in this work.

$(CA_1)$ Suppose for $\aleph_1, \aleph_2 \in \Psi$ there exist a constants $\omega_{g_1}, \omega_{g_2} \in [0, 1)$ such that

$$|g_1(\aleph_1) - g_1(\aleph_2)| \leq \omega_{g_1}|\aleph_1 - \aleph_2|, \quad |g_2(\aleph_1) - g_2(\aleph_2)| \leq \omega_{g_2}|\aleph_1 - \aleph_2|.$$

$(CA_2)$ Let for $\aleph \in \Psi$, we have few constants $\chi_{g_1}, \mu_{g_1}, \chi_{g_2}, \mu_{g_2} \in [0, 1)$ such that

$$|g_1(\aleph)| \leq \chi_{g_1}|\aleph| + \mu_{g_1}, \quad |g_2(\aleph)| \leq \chi_{g_2}|\aleph| + \mu_{g_2}.$$

$(CA_3)$ Let for $\aleph \in \Psi$, we have different constants $D_f, W_f \in [0, 1)$ so that

$$|f((t, \aleph(t), \aleph(\lambda(t)), \aleph(\tau - t))| \leq D_{f_0} + D_{f_1}|\aleph| + D_{f_2}\lambda|\aleph| + D_{f_3}\tau|\aleph|.$$

$$|f((t, \aleph(t), \aleph(\lambda(t)), \aleph(\tau - t))| \leq D_f + W_f\aleph$$

where $D_f = D_{f_0}$ and $W_f = D_{f_1} + D_{f_2}\lambda + D_{f_3}\tau$

$(CA_4)$ Let for $\aleph \in R$, we have some constants $D_1, W_1, D_2, W_2 \in [0, 1)$ such that

$$|I_j(\aleph)| \leq D_1|\aleph| + W_1, |J_j(\aleph)| \leq D_2|\aleph| + W_2.$$

$(CA_5)$ Let $I_i, J_i : R \to R$ and we have few constant $K_I^i, K_J^i \in [0, \frac{1}{m})$ such that

$$|I_i(\aleph_1) - I_i(\aleph_2)| \leq K_I^i|\aleph_1 - \aleph_2|, \ |J_i(\aleph_1) - J_i(\aleph_2)| \leq K_J^i|\aleph_1 - \aleph_2|$$

and

$$|f((t, \aleph_1(t), \aleph_2(\lambda(t)), \aleph_2(\tau - t))| - f((t, \aleph_2(t), \aleph_2(\lambda(t)), \aleph_2(\tau - t))|,$$

$\leq L_f|\aleph_1 - \aleph_2| + L_f|\aleph_1(\lambda t) - \aleph_2(\lambda t)| + L_f|\aleph_1(t - \tau) - \aleph_2(t - \tau)| \ \leq K_\aleph|\aleph_1 - \aleph_2|$ for all $\aleph_1, \aleph_2 \in R$ and $i = 1, 2, 3, \ldots$.

A function $\aleph \in \Psi$ with its $\alpha$- derivative exist on $[0, \tau] - \{t_1, t_2, t_3, \ldots, t_m, \}$ it is considered the solution to the given fractional impulsive problem with boundary conditions if it satisfies the equation (1) under consideration.

**Lemma 3.** *Let $\aleph \in \Psi$ represent the solution to the impulsive fractional problem, where $\sigma$ is a function in $C([0, 1], R)$ defined as follows*

$$\begin{cases} D^\alpha\aleph(t) = \sigma(t), , t \neq t_i, \ 1 < \alpha \leq 2, \\ \triangle\aleph(t_j) = I_j(\aleph(t_j)), \triangle\aleph^{'}(t_j) = J_j(\aleph(t_j)), j = 1, 2, 3, \ldots, l \\ \triangle_1\aleph(0) + \triangle_2\aleph(t) = g_1(\aleph), \triangle_3\aleph^{'}(0) + \triangle_4\aleph^{'}(\tau) = g_2(\aleph), \end{cases} \tag{2}$$

*if and only if $\aleph$ is the answer of the impulsive fractional integral equation as*

$$\begin{aligned} \aleph(t) = & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta + \frac{\triangle_1}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta \\ & + \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big) \\ & \times \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\ & + \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{\tau}(t-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\ & + \frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k}I_j(\aleph(t_j)) + \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) \\ & - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\sum_{j=1}^{k}J_j(\aleph(t_j))\Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)g_2(\aleph) \\ & + \frac{g_1(\aleph)}{\triangle_1+\triangle_2} - \frac{\triangle_2}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_k-1}^{\tau}(\tau-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta. \end{aligned} \tag{3}$$

*Proof.* Let $\aleph$ be the solution to (2). Then, for $\sigma \in C([0,1],R)$ and $t \in [0,\tau]$, and by applying Lemma 2 to the given problem, we obtain two constants, $b_0$ and $b_1$, such that

$$\begin{aligned}\aleph(t) &= I^{\alpha}\sigma(t) - b_0 - b_1 t,\ t \in [0,t_1],\\ \aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta - b_0 - b_1 t,\ t \in [0,t_1].\end{aligned} \tag{4}$$

After differentiating, we have

$$\aleph'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta - b_1,\ t \in [0,t_1]. \tag{5}$$

Similarly for $t \in (t_1,t_2]$, there are constants $\mathbf{q}_0, \mathbf{q}_1$ with

$$\begin{aligned}\aleph(t) &= I^{\alpha}\sigma(t) - \mathbf{q}_0 - \mathbf{q}_1(t-t_1),\ t \in (t_1,t_2],\\ \aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta - \mathbf{q}_0 - \mathbf{q}_1(t-t_1),\ t \in [0,t_1].\end{aligned} \tag{6}$$

After differentiating, we get

$$\aleph'(t) = \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta - \mathbf{q}_1,\ t \in (t_1,t_2], \tag{7}$$

and

$$\begin{aligned}\aleph(t_1^-) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1} - b_0 - b_1 t_1,\ t \in [0,t_1],\\ \aleph(t_1^+) &= -b_0,\end{aligned} \tag{8}$$

$$\begin{aligned}\aleph'(t_1^-) &= \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta - \mathbf{q}_1,\ t \in (t_1,t_2],\\ \aleph'(t_1^+) &= -\mathbf{q}_1,\end{aligned} \tag{9}$$

Now we are applying the conditions of impulsive after simplification we get the following

$$\begin{aligned}-\mathbf{q}_0 &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta - b_0 - b_1 t_1 + I_1(\aleph(t_1)),\\ -\mathbf{q}_1 &= \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta - b_0 + J_1(\aleph(t_1)).\end{aligned} \tag{10}$$

Putting the values of $\mathbf{q}_0$ and $\mathbf{q}_1$ in given equation we get

$$\begin{aligned}\aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta\\ &+ \frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta + (t-t_1)J_1(\aleph(t_1)) + I_1(\aleph(t_1)) - b_0 - b_1 t.\end{aligned} \tag{11}$$

Generally for $t \in (t_{j-1},t_j]$, reiterating the same procedure we have

$$\begin{aligned}\aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_1-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta\\ &+ \sum_{j=1}^{k}(t-t_j)\frac{1}{\Gamma(\alpha-1)}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta + \sum_{j=1}^{k}(t-t_j)J_j(\aleph(t_j))\\ &+ \sum_{j=1}^{k}I_j(\aleph(t_j)) - b_0 - b_1 t.\end{aligned} \tag{12}$$

After evaluating and simplifying the BCs, we have the following for the constant $b_0, b_1$,

$$
\begin{aligned}
b_0 &= \frac{\triangle_2}{\triangle_1+\triangle_2}\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta \\
&+ \sum_{j=1}^{k}(\tau-t_j)\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-2}\frac{1}{\Gamma(\alpha)}\sigma(\zeta)d\zeta+\sum_{j=1}^{k}(\tau-t_j)J_j(\aleph(t_j))+\sum_{j=1}^{k}(\tau-t_j)I_j(\aleph(t_j))\Big) \\
&- \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big(\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta+\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta\Big) \\
&- \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\sum_{j=1}^{k}J_j(\aleph(t_j))+\Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big)g_2(\aleph(t_j))-\Big(\frac{1}{\triangle_1+\triangle_2}\Big)g_1(\aleph(t_j)), \\
b_1 &= \frac{\triangle_4}{\triangle_4+\triangle_3}\Big(\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta+\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \sum_{j=1}^{k}J_j(\aleph(t_j))\Big)-\Big(\frac{1}{\triangle_4+\triangle_3}\Big)g_2(\aleph(t_j)).
\end{aligned}
$$

By substituting these values into (12), we obtain the desired solution as expressed in (3)

$$
\begin{aligned}
\aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta \\
&- \frac{\triangle_2}{\triangle_1+\triangle_2}\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(\tau-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta+\sum_{j=1}^{k}(t-t_j)J_j(\aleph(t_j)) \\
&+ \sum_{j=1}^{k}I_j(\aleph(t_j))+\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_j-1}^{t_j}(t-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-1}\sigma(\zeta)d\zeta+\sum_{j=1}^{k}(t-t_j)\frac{1}{\Gamma(\alpha-1)}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \sum_{j=1}^{k}(t-t_j)J_j(\aleph(t_j)) \\
&+ \sum_{j=1}^{k}I_j(\aleph(t_j))\Big)+\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big(\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{T}(\tau-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \sum_{j=1}^{k}J_j(\aleph(t_j))+\frac{g_2(\aleph)}{\triangle_4}\Big)-\frac{t\triangle_4}{\triangle_4+\triangle_3}\Big(\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(\tau-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-2}\sigma(\zeta)d\zeta \\
&+ \sum_{j=1}^{k}J_j(\aleph(t_j))\Big)+\Big(\frac{t}{\triangle_4+\triangle_3}\Big)g_2(\aleph)+\Big(\frac{1}{\triangle_1+\triangle_2}\Big)g_1(\aleph).
\end{aligned}
$$

By modifying the term, we can derive the result in (3). Additionally, assuming that $\aleph$ is a solution to the given problem, a straightforward calculation leads to the answer in (3), which satisfies (13),

**Corollary 1.** *The solution to the problem in* (1) *is provided by*

$$\begin{aligned}
\aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\zeta)^{\alpha-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta + \frac{\triangle_1}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta \\
&- \frac{\triangle_2}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\int_{t_k}^{t}(\tau-\zeta)^{\alpha-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta + \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4 t}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(t-t_j) \\
&- \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-2}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta + \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\alpha-1)} \\
&\times \int_{t_k}^{t}(\tau-\zeta)^{\alpha-2}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta + \frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k}I_j(\aleph(t_j)) + \Big(\sum_{j=1}^{k}(t-t_j) \\
&+ \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\sum_{j=1}^{k}J_j(\aleph(t_j)) \\
&+ \Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)g_2(\aleph) + \frac{g_1(\aleph)}{\triangle_1+\triangle_2}.
\end{aligned} \tag{13}$$

Using Corollary 1, we change the problem (13) into a fixed-point problem, where $\aleph = t(\aleph)$. Based on this, we develop results for solving the problem. To prove existence and uniqueness, we define five operators to help establish the main results, as follows.

$$\begin{aligned}
E_0 &: \Psi \to \Psi \\
E_0\aleph(t) &= \frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k}I_j(\aleph(t_j)) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\sum_{j=1}^{k}J_j(\aleph(t_j)) + \Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)g_2(\aleph) \\
&+ \frac{g_1(\aleph)}{\triangle_1+\triangle_2} + \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j).
\end{aligned}$$

We define the operators for integral part as:

$$\begin{aligned}
E_1 &: \Psi \to \Psi \\
E_1\aleph(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\zeta)^{\alpha-1}f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta - \frac{\triangle_2}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\int_{t_k}^{t}(\tau-\zeta)^{\alpha-1}f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(t-\zeta))d\zeta, \\
E_2 &: \Psi \to \Psi \\
E_2\aleph(t) &= \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big) \\
&\times \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-2}f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta, \\
E_3 &: \Psi \to \Psi \\
E_3\aleph(t) &= \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-2}f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta, \\
E_4 &: \Psi \to \Psi \\
E_4\aleph(t) &= \frac{\triangle_1}{(\triangle_1+\triangle_2)\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\alpha-1}f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta.
\end{aligned}$$

Let $\tau : \Psi \to \Psi$, then $\tau$ is defined by

$$\tau\aleph(t) = E_0\aleph(t) + E_1\aleph(t) + E_2\aleph(t) + E_3\aleph(t) + E_4\aleph(t).$$

Hence, investigating the answer to the provided (13) problem is similar to investigate fixed point for the operator $\tau$.

**Theorem 1.** *The operator $E_0 : \Psi \to \Psi$ is Lipschitz continuous with a constant $K_I = \sum_{i=1}^{m} K^i I \in [0,1)$. As a result, $E_0$ is also $\aleph$-Lipschitz with the same constant $K^i I \in [0,1)$. Additionally, $E_0$ satisfies the following relation*

$$\|E_0\aleph\| \leq U + V\|Y\|, \tag{14}$$

*where*

$$\begin{aligned} U &= \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| W_1 + \left| l\tau + \frac{\triangle_2\triangle_4 t}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| W_2 + \left|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{\tau}{\triangle_3+\triangle_4}\right|\mu_{g_2} \\ &+ \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right|\mu_{g_1} \end{aligned} \tag{15}$$

*and*

$$\begin{aligned} V &= \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| D_1 + \left| l\tau + \frac{\triangle_2\triangle_4 t}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| D_2 + \left|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{\tau}{\triangle_3+\triangle_4}\right|\chi_{g_2} \\ &+ \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right|\chi_{g_1}. \end{aligned} \tag{16}$$

*Proof.* Using $(CA_1)$ and $(CA_4)$, we have

$$\begin{aligned} \sup_{t\in[0,\tau]} |E_0\aleph_1 - E_0\aleph_2| &= \sup_{t\in[0,\tau]} \left| \frac{\triangle_1}{\triangle_1+\triangle_2} \sum_{j=1}^{k} \Big( I_j(\aleph_1(t_j)) - I_j(\aleph_2(t_j)) \Big) + \Big( \sum_{j=1}^{k} (t-t_j) \right. \\ &+ \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} \\ &- \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau - t_j) - \frac{t\triangle_4}{(\triangle_3+\triangle_4)}\Big)\Big(J_j(\aleph_1(t_j) - J_j(\aleph_2(t_j))\Big) \\ &+ \Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)\Big(g_2(\aleph_1) - g_2(\aleph_2)\Big) + \frac{1}{\triangle_1+\triangle_2}\Big(g_1(\aleph_1) \\ &- \left. g_1(\aleph_2)\Big)\right|. \end{aligned}$$

That is the result of further simlification

$$\begin{aligned} \|E_0\aleph_1 - E_0\aleph_2\| &\leq \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| K_1^i \|\aleph_1 - \aleph_2\| + \left| l\tau + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| K_j^i \|\aleph_1 - \aleph_2\| \\ &+ \left|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{\tau}{\triangle_3+\triangle_4}\right|\omega_{g_2}\|\aleph_1 - \aleph_2\| + \left|\frac{1}{\triangle_1+\triangle_2}\right|\omega_{g_1}\|\aleph_1 - \aleph_2\|\Big). \end{aligned}$$

Hence one has

$$\begin{aligned} \|E_0\aleph_1 - E_0\aleph_2\| &\leq \Big(\left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| K_1^i + \left| l\tau + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| K_j^i + \left|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{\tau}{\triangle_3+\triangle_4}\right|\omega_{g_2} \\ &+ \left|\frac{1}{\triangle_1+\triangle_2}\right|\omega_{g_1}\Big) \times \|\aleph_1 - \aleph_2\|. \end{aligned} \tag{17}$$

Using

$$\begin{aligned} K_1 &= \Big(\left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| K_1^i + \left| l\tau + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| K_j^i + \left|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{\tau}{\triangle_3+\triangle_4}\right|\omega_{g_2} \\ &+ \left|\frac{1}{\triangle_1+\triangle_2}\right|\omega_{g_1}\Big). \end{aligned}$$

then (17) becomes

$$\|E_0\aleph_1 - E_0\aleph_2\| \leq K_1\|\aleph_1 - \aleph_2\|.$$

Thus For constant $K \in [0,1) E_0$ is Lipschitz
Using $(CA_2)$ and $(CA_5)$,for growth relation we get the following

$$\begin{aligned}||E_0\aleph|| &= \sup_{t\in[0,\tau]}\Bigg|\frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k} I_j(\aleph_1(t_j)+\Big(\sum_{j=1}^{k}(t-t_j)+\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}-(\frac{\triangle_2}{\triangle_1+\triangle_2})\sum_{j=1}^{k}(t-t_j)\\ &- \frac{t\triangle_4}{(\triangle_3+\triangle_4)}\Big)J_j(\aleph(t_j)+\Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}+\frac{t}{\triangle_3+\triangle_4}\triangle_2\Big)g_2(\aleph)+\Big(\frac{1}{\triangle_1+\triangle_2}\Big)g_1(\aleph)\Bigg|,\end{aligned}$$

$$\begin{aligned}||E_0\aleph|| &\leq \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big|(D_1||\aleph||+W_1)+\Big|l\tau+\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big|(D_2||\aleph||+W_2)+\Big|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\\ &+ \frac{\tau}{\triangle_3+\triangle_4}\Big|(\chi_{g_2}||\aleph||+\mu_{g_2})+\Big|\frac{1}{\triangle_1+\triangle_2}\Big|(\chi_{g_1}||\aleph||+\mu_{g_1}),\end{aligned} \tag{18}$$

then (18), becomes

$$|||E_0\aleph(t)||| \leq U+V||\aleph||,$$

where $U$ and $V$ are given in (15) and (16).

**Lemma 4.** *The operator*

$$E_1\aleph(t):\Psi\to\Psi$$

*is continuous and satisfies the following growth condition*

$$||E_1\aleph|| \leq \Big(1+|\frac{\triangle_2}{(\triangle_1+\triangle_2)}|\Big)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}(D_f+W_f||\aleph||). \tag{19}$$

*Proof.* Let $\aleph_n$ be a sequence in $B_k=\{\aleph\in\aleph : ||\aleph||\leq r\}$ such that $\aleph_n\to\aleph$ as $n\to\infty$.
This implies that

$$\frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}(h(\zeta,\aleph_n)-h(\zeta,\aleph))\to 0, n\to\infty,$$

and

$$\frac{(\tau-\zeta)^{\alpha-1}}{\Gamma(\alpha)}(h(\zeta,\aleph_n)-h(\zeta,\aleph))\to 0, n\to\infty.$$

By applying the Lebesgue Dominated Convergence Theorem, it can be observed that
$||E_1(\aleph_n)-E_1(\aleph)||\to 0$ as $n\to\infty$. This suggests that $E_1$ is a continuous function.
Using $(CA_3)$, for growth relation then

$$|E_1(\aleph(t))| \leq \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\zeta)^{\alpha-1}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta+\Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-1}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(t-\zeta))|d\zeta,$$

$$\begin{aligned}\sup_{t\in[0,\tau]}|E_1(\aleph(t))| &\leq \sup_{t\in[0,\tau]}\Big(\frac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big|\frac{(\tau-t_k)^{\alpha}}{\Gamma(\alpha+1)}\Big)(D_f+W_f||\aleph||),\\ ||E_1\aleph|| &\leq \Big(1+\Big||\frac{\triangle_2}{\triangle_1+\triangle_2}\Big||\Big)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}(D_f+W_f||\aleph||).\end{aligned}$$

**Theorem 2.** *The operator$E_1:\aleph\to\aleph$ is defined as compact and satisfies the $\aleph-$ Lipschitz condition with a zero constant.*

*Proof.* Clearly $E_1$ satisfies growth condition so $E_1$ is bounded on $B_k=\{\aleph\in\aleph : ||\aleph||\leq r\}$ .
Let $\aleph\in B_k$, we have

$$||E_1\aleph|| \leq \Big(1+\Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big|\Big)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}(D_f+W_f||\aleph||)\leq Q_1.$$

So $E_1$ is bounded.
We learn more $0 \leq t_1 \leq t_2 \leq \tau$, we show that $E_1$ is equi-continuous.

$$
\begin{aligned}
|E_1(\aleph(t_1)) - E_1(\aleph(t_2))| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_1}(t_1-\zeta)^{\alpha-1}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta \\
&- \frac{\triangle_2}{\triangle_1+\triangle_2}\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-1}|h(\zeta,\aleph(\zeta))|d\zeta \\
&+ \frac{\triangle_2}{\triangle_1+\triangle_2}\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-1}|h(\zeta,\aleph(\zeta))|d\zeta \\
&- \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_2}(t_2-\zeta)^{\alpha-1}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta.
\end{aligned}
$$

One has

$$
|E_1(\aleph(t_1)) - E_1(\aleph(t_2))| \leq \frac{(D_f + W_f\|\aleph\|)}{\Gamma(\alpha+1)}\Big((t_1-t_k)^{\alpha} - (t_2-t_k)^{\alpha}\Big).
$$

As $t_1 \to t_2$, then $|E_1(\aleph(t_1)) - E_1(\aleph(t_2))| \to 0$. Hence $E_1$ is equicontinuous so $E_1$ is compact.
then by Proposition **??**, $E_1$ is $\aleph-$ Lipschitz with constant zero.

**Lemma 5.** *The operator*

$$
E_2 : \Psi \to \Psi
$$

*is continuous and satisfies the following growth condition*

$$
\|E_2\aleph\| \leq \Big(l + |\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}|\Big)\frac{\tau^{\alpha+1}}{\Gamma(\alpha)}(D_f + W_f\|\aleph\|). \tag{20}
$$

*Proof.* Let $\aleph_n$ be a sequence in $B_k = \{\aleph \in \aleph : \|\aleph\| \leq r\}$ such that $\aleph_n \to \aleph$ as $n \to \infty$.
this implies that

$$
\frac{(t_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}(h(\zeta,\aleph_n) - h(\zeta,\aleph)) \to 0, n \to \infty,
$$

By applying the Lebesgue Dominated Convergence Theorem, it can be observed that

$$
\|E_2(\aleph_n) - E_2(\aleph)\| \to 0 \text{ as } n \to \infty.
$$

This suggests that $E_2$ is continuous..
Additionally, for the growth relation using $(CA_3)$, the following holds

$$
\begin{aligned}
|E_2(\aleph(t))| &\leq \left|\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\right| \\
&\times \sum_{j=1}^{k}\int_{t_j-1}^{t_j}\frac{(t_j-\zeta)^{\alpha-2}}{\Gamma(\alpha)}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta,
\end{aligned}
$$

which yields

$$
\sup_{t\in[0,\tau]}|E_2(\aleph(t))| \leq \sup_{t\in[0,\tau]}\left|\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \int_{t_j-1}^{t_j}\frac{(t_j-\zeta)^{\alpha-2}}{\Gamma(\alpha)}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta.
$$

thus one has

$$
\|E_2(\aleph)\| \leq \left|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \frac{\tau^{\alpha+1}}{\Gamma(\alpha)}(D_f + W_f\|\aleph\|).
$$

Which is the required relation.

**Lemma 6.** *The operator $E_2 : \aleph \to \aleph$ is compact and $\aleph$-Lipschitz with a zero constant.*

*Proof.* Similarly, it can be derived as shown in the Lemma 5.

**Lemma 7.** *The operator*

$$E_3 : \Psi \to \Psi$$

*is continuous and meets the following growth condition*

$$\|E_3\aleph\| \leq \left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \frac{\tau^{\alpha+1}}{\Gamma(\alpha)}(D_f + W_f\|\aleph\|). \tag{21}$$

*Proof.* Let $\aleph_n$ be a sequence in $B_k = \{\aleph \in \aleph : \|\aleph\| \leq r\}$ such that $\aleph_n \to \aleph$ as $n \to \infty$.
this implies that

$$\frac{(\tau-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}(h(\zeta,\aleph_n) - h(\zeta,\aleph)) \to 0, n \to \infty.$$

Using Lebesgue Dominated convergence theorem

$$\|E_3(\aleph_n) - E_3(\aleph)\| \to 0 \text{ as } n \to \infty.$$

This means that $E_3$ is continuous
For the growth relation using $(CA_3)$, then

$$|E_3(\aleph(t))| \leq \left|\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\right| \times \frac{1}{\alpha(\alpha-1)}\int_{t_k}^{\tau}(\tau-\zeta)^{\alpha-2}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta,$$

$$\sup_{t\in[0,\tau]}|E_3(\aleph(t))| \leq \sup_{t\in[0,\tau]}\left|\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \int_{t_k}^{t}(\tau-\zeta)^{\triangle_1-2}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta.$$

Hence one has

$$\|E_3(\aleph)\| \leq \left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \frac{\tau^{\alpha+1}}{\Gamma(\alpha)}(D_f + W_f\|\aleph\|).$$

Which is the required relation.

**Lemma 8.** *The defined operator as, $E_3 : \aleph \to \aleph$ is compact and $E_3$ is $\aleph$– Lipschitz with zero constant .*

*Proof.* As proof is simple, we have omitted it.

**Lemma 9.**

$$E_4 : \Psi \to \Psi$$

*as the operator is continuous and satisfy the following growth condition*

$$\|E_4\aleph\| \leq \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| \times \frac{l\tau^{\alpha}}{\Gamma(\alpha+1)}(D_f + W_f\|\aleph\|). \tag{22}$$

*Proof.* Let $\aleph_n$ be a sequence in $B_k = \{\aleph \in \aleph : \|\aleph\| \leq r\}$ such that $\aleph_n \to \aleph$ as $n \to \infty$.
this implies that

$$\frac{(t_j-\zeta)^{\alpha-1}}{\Gamma(\alpha)}(h(\zeta,\aleph_n) - h(\zeta,\aleph)) \to 0, n \to \infty,$$

Using Lebesgue Dominated convergence theorem
$\|E_4(\aleph_n) - E_4(\aleph)\| \to 0$ as $n \to \infty$. this shows that $E_4$ is continuous.
For the growth relation with $(CA_3)$, we have

$$|E_4(\aleph(t))| \leq \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right|\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j-1}^{t_j}(t_j-\zeta)^{\alpha-1}|f((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))|d\zeta,$$

$$\sup_{t\in[0,t]} |E_4(\aleph(t))| \leq \sup_{t\in[0,t]} \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| \times \frac{(t_j - t_{j-1})^\alpha}{\Gamma(\alpha+1)} \times (D_f + W_f\|\aleph\|),$$

$$\|E_4\aleph\| \leq \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| \times \frac{l\tau^\alpha}{\Gamma(\alpha+1)}(D_f + W_f\|\aleph\|).$$

This is the needed relation.

**Lemma 10.** *The operator $E_4 : \aleph \to \aleph$ is compact and $\aleph$-Lipschitz with a zero constant.*

*Proof.* As proof is simple, we have omitted it.

Next we demonstrate that the four operators $E_1, E_2, E_3$ and $E_4$ are combined,meet the growth condition and are continuous. Finally, we show that the four operators $E_1, E_2, E_3$ and $E_4$ are compact and $\aleph-$ Lipschitz with zero constants.

**Theorem 3.** *The operators $E_1, E_2, E_3$ and $E_4 : \Psi \to \Psi$, are continuous and satisfy the following relation*

$$\begin{aligned} \|E_1\aleph\| + \|E_2\aleph\| + \|E_3\aleph\| + \|E_4\aleph\| &\leq \left(1 + \left|\frac{\triangle_2}{(\triangle_1+\triangle_2)}\right|\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \left|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right|\frac{\tau^{\alpha+1}}{\Gamma(\alpha)}\right. \\ &+ \left.\left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right| \times \frac{\tau^{\alpha+1}}{\Gamma(\alpha)} + \left|\frac{\triangle_1}{(\triangle_1+\triangle_2)}\right| \times \frac{l\tau^\alpha}{\alpha(\alpha+1)}\right) \\ &\times (D_f + W_f\|\aleph\|). \end{aligned} \tag{23}$$

*$E_1, E_2, E_3$ and $E_4$, are all continuous,hence $E_1 + E_2 + E_3 + E_4$, is also continuous.*

**Lemma 11.** *The operators $E_1, E_2, E_3$ and $E_4 : \Psi \to \Psi$, are $\aleph-$ Lipschitz and compact with zero constant.*

*Proof.* Since $E_1, E_2, E_3$ and $E_4$, are compact so $E_1 + E_2 + E_3 + E_4$, is also compact,hence according to Proposition **??**, It is $\aleph$-Lipschitz with a constant value of zero.

**Theorem 4.** *Assuming that $(CA_1), (CA_2), (CA_3)$, and $(CA_4)$ hold, the problem 3 has at least one solution $\aleph \in \Psi$, and the solution set is bounded in $\Psi$.*

*Proof.* Let the operators $E_0, E_1, E_2, E_3$, and $E_4$, along with t : $\Psi \to \Psi$, be defined in the previous section. They are continuous and bounded. Moreover, $E_0$ is $\aleph$-Lipschitz with a constant $K \in [0, 1)$, while $E_1, E_2, E_3$, and $E_4$ are $\aleph$-Lipschitz with zero constants. Let

$$H = \{\aleph \in \Psi : \text{there exist } \lambda \in [0, 1] \text{ such that } \aleph = \lambda\tau\aleph\}.$$

We aim to show that $H$ is bounded in $\Psi$. Let $\aleph \in H$, $\lambda \in [0, 1]$ such that $\|\aleph\| = \lambda\|\tau\aleph\|$. It follows from (14) and (23) that

$$\|\aleph\| \leq |\lambda|\Big(\|E_0\aleph\| + \|E_1\aleph\| + \|E_2\aleph\| + \|E_3\aleph\| + \|E_4\aleph\|\Big).$$

$$\begin{aligned} \|\aleph\| &\leq |\lambda|\left[U + V\|\aleph\| + \left(1 + \left|\frac{\triangle_2}{\triangle_1+\triangle_2}\right|\right)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \left(l + \left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right|\right)\frac{\tau^{\alpha+1}}{\Gamma(\alpha)}\right. \\ &+ \left.\left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right|\frac{\tau^{\alpha+1}}{\Gamma(\alpha)} + \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| \times \frac{l\tau^\alpha}{\Gamma(\alpha+1)}\right](W_f\|\aleph\| + D_f). \end{aligned}$$

This inequality shows that $H$ is bounded. If not, assume that $\zeta = |\aleph| \to \infty$. By dividing both sides of the inequality by $|\aleph|$

$$\begin{aligned} 1 &\leq |\lambda|\left[\frac{U + V\|\aleph\| + \left(1 + \left|\frac{\triangle_2}{\triangle_1+\triangle_2}\right|\right)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \left(l + \left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right|\right)\frac{\tau^{\alpha+1}}{\Gamma(\alpha)}}{\zeta}\right. \\ &+ \left.\frac{\left|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\right|\frac{\tau^{\alpha+1}}{\Gamma(\alpha)} + \left|\frac{\triangle_1}{\triangle_1+\triangle_2}\right| \times \frac{l\tau^\alpha}{\Gamma(\alpha+1)}}{\zeta}\right] \times (W_f\|\aleph\| + D_f). \end{aligned}$$

taking limit as $\zeta \to \infty$, we get the following relation.
$1 \leq 0$, which is a contradiction. Therefore, $\tau$ must be bounded in $\Psi$ and have at least one fixed point.

**Theorem 5.** *Assuming the hypothesis* $(CA_5)$ *and that* $\mathbf{K} < 1$, *where*

$$\begin{aligned}\mathbf{K} &= \Big[\Big(\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big| \times \frac{l\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big| \times \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{l\tau^{\alpha+1}}{\Gamma(\alpha)} \\ &+ \Big|\frac{\triangle_2\triangle_4}{(r\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{\tau^\alpha}{\Gamma(\alpha)}\Big)K_\aleph + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big|K_I^i + \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big|K_J^i + \Big|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} \\ &+ \frac{1}{\triangle_3+\triangle_4}\Big|\omega_{g_2} + \Big|\frac{1}{\triangle_1+\triangle_2}\Big|\omega_{g_1}\Big].\end{aligned}$$

*Then the problem* (1) *under considration has unique solution.*

*Proof.* Assuming there are $\aleph_1, \aleph_2 \in \Psi$ be two solutions of the given problem (1), then

$$\begin{aligned}\|\tau\aleph_1 - \tau\aleph_2\| &\leq \Big[\frac{K_\aleph\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big| \times \frac{K_\aleph l\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big| \times \frac{K_\aleph\tau^\alpha}{\Gamma(\alpha+1)} \\ &+ \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{K_\aleph l\tau^{\alpha+1}}{\Gamma(\alpha)} + \Big|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{K_\aleph\tau^\alpha}{\Gamma(\alpha)} \\ &+ \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big|K_I^i + \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big|K_J^i + \Big|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} \\ &+ \frac{1}{\triangle_3+\triangle_4}\Big|\omega_{g_2} + \Big|\frac{1}{\triangle_1+\triangle_2}\Big|\omega_{g_1}\Big]\|\aleph_1 - \aleph_2\|\end{aligned}$$

$$\begin{aligned}\|\tau\aleph_1 - \tau\aleph_2\| &\leq \Big[\Big(\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big| \times \frac{l\tau^\alpha}{\Gamma(\alpha+1)} + \Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big| \times \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \Big|l \\ &+ \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{l\tau^{\alpha+1}}{\Gamma(\alpha)} + \Big|\frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{\tau^\alpha}{\Gamma(\alpha)}\Big)K_\aleph \\ &+ \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big|K_I^i + \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big|K_J^i + \Big|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} \\ &+ \frac{1}{\triangle_3+\triangle_4}\Big|\omega_{g_2} + \Big|\frac{1}{\triangle_1+\triangle_2}\Big|\omega_{g_1}\Big]\|\aleph_1 - \aleph_2\|.\end{aligned}$$

Thus, $\tau$ is a contraction mapping, and by the Banach fixed-point theorem, $\tau$ has a unique fixed point. Therefore, the problem has a unique solution.

## 4. STABILITY ANALYSIS

Establish stability results for the obtained outcome, with stability based on the U-H concept this stability is determined about the best approximate or exact solution of the problem.

**Remark 1.** *Let for* $\epsilon > 0$ *and we have some independent mapping say* $\psi, v_i, v_j$ *such that*

$u_1$ : $|\psi(t)| \leq \epsilon, |v_i(t)| \leq \epsilon$ *and* $|v_j(t)| \leq \epsilon$, *where each* $t \in [0, 1]$ *and i, j=1,2,3, . . . , m.*

$u_2$ : $D^\xi\aleph(t) = \Xi(t, \aleph(t), \aleph(\lambda t), \aleph(\tau - t)) + \psi(t), t \neq t_i, t_j$

$u_3$ : $\triangle\aleph(t_i) = I_i(\aleph(t_j)) + v_i(t)$ *and* $\triangle\aleph'(t_i) = J_j(\aleph(t_i)) + v_j(t)$

$u_4$ : *We use the inequalities* $|g_1(\aleph)| \leq \chi_{g_1}|\aleph| + \mu_{g_1} \leq \epsilon$ *and* $|g_2(\aleph)| \leq \chi_{g_2}|\aleph| + \mu_{g_2} \leq \epsilon$ *for the easiness.*

**Remark 2.** *The answer of the problem for* $\aleph \in \Psi$ *to*

$$\begin{cases} D^\xi\aleph(t) = \Xi(t, \aleph(t), \aleph(\lambda t), \aleph(\tau - t)) + \psi(t), t \neq t_i, \ 1 < \xi \leq 2, \\ \triangle\aleph(t_i) = I_i(\aleph(t_j)) + v_i, \triangle\aleph'(t_i) = J_j(\aleph(t_i)) + v_j, \\ \triangle_1\aleph(0) + \triangle_2\aleph(1) = g_1(\aleph), \triangle_3\aleph'(0) + \triangle_4\aleph'(1) = g_2(\aleph), \end{cases} \tag{24}$$

*with the help of Lemma 2, we have the following relation for problem* (6), *as*

$$\begin{aligned}
\aleph(t) &= F(\aleph(t)) + \frac{1}{\Gamma(\xi)}\int_{t_k}^{t}(t-\zeta)^{\xi-1}\psi(\zeta))d\zeta \\
&+ \frac{1}{(\triangle_1+\triangle_2)}\Big(\frac{\triangle_1}{\Gamma(\xi)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\xi-1}\psi(\zeta))d\zeta \\
&- \frac{\triangle_2}{\Gamma(\xi)}\int_{t_k-1}^{t}(\tau-\zeta)^{\xi-1}\psi(\zeta))d\zeta\Big) \\
&+ \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) \\
&- \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\times\frac{1}{\Gamma(\xi-1)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\xi-2}\psi(\zeta))d\zeta \\
&+ \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\xi-1)} \\
&\times\int_{t_k}^{t}(\tau-\zeta)^{\xi-2}\psi(\zeta))d\zeta + \frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k}v_i(\aleph(t_i)) + \Big(\sum_{j=1}^{k}(t-t_j) \\
&+ \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\sum_{j=1}^{k}v_j(\aleph(t_j)) \\
&+ \Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)g_2(\aleph) + \frac{g_1(\aleph)}{\triangle_1+\triangle_2}.
\end{aligned} \tag{25}$$

*Where* $F(\aleph(t))$ *is given in following*

$$\begin{aligned}
F(\aleph(t)) &= \frac{1}{\Gamma(\xi)}\int_{t_k}^{t}(t-\zeta)^{\xi-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta \\
&+ \frac{1}{(\triangle_1+\triangle_2)}\Big(\frac{\triangle_1}{\Gamma(\xi)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\xi-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta \\
&- \frac{\triangle_2}{\Gamma(\xi)}\int_{t_k-1}^{t}(\tau-\zeta)^{\xi-1}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta\Big) \\
&+ \Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) \\
&- \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\times\frac{1}{\Gamma(\xi-1)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-\zeta)^{\xi-2}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta \\
&+ \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\xi-1)} \\
&\times\int_{t_k}^{t}(\tau-\zeta)^{\xi-2}\Xi((\zeta,\aleph(\zeta),\aleph(\lambda(\zeta)),\aleph(\tau-\zeta))d\zeta + \frac{\triangle_1}{\triangle_1+\triangle_2}\sum_{j=1}^{k}I_j(\aleph(t_j)) + \Big(\sum_{j=1}^{k}(t-t_j) \\
&+ \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\sum_{j=1}^{k}J_j(\aleph(t_j)) \\
&+ \Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)g_2(\aleph) + \frac{g_1(\aleph)}{\triangle_1+\triangle_2}.
\end{aligned} \tag{26}$$

From (25), we can write for $t \in [t_{j-1}, t_j]$ one has

$$\begin{aligned}
&\Big| \aleph(t) - F(\aleph(t)) - \frac{1}{\Gamma(\xi)} \int_{t_k}^{t} (t-\zeta)^{\xi-1} \psi(\zeta)) d\zeta \\
&- \frac{1}{(\triangle_1 + \triangle_2)} \Big( \frac{\triangle_1}{\Gamma(\xi)} \sum_{j=1}^{k} \int_{t_j - 1}^{t_j} (t_j - \zeta)^{\xi-1} \psi(\zeta)) d\zeta \\
&+ \frac{\triangle_2}{\Gamma(\xi)} \int_{t_k - 1}^{t} (\tau - \zeta)^{\xi-1} \psi(\zeta)) d\zeta \Big) \\
&- \Big( \sum_{j=1}^{k} (t - t_j) + \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{\triangle_2}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} (\tau - t_j) \\
&- \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \times \frac{1}{\Gamma(\xi - 1)} \sum_{j=1}^{k} \int_{t_j - 1}^{t_j} (t_j - \zeta)^{\xi-2} \psi(\zeta)) d\zeta \\
&- \Big( \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \frac{1}{\Gamma(\xi - 1)} \\
&\times \int_{t_k}^{t} (\tau - \zeta)^{\xi-2} \psi(\zeta)) d\zeta - \frac{\triangle_1}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} v_i(\aleph(t_i)) - \Big( \sum_{j=1}^{k} (t - t_j) \\
&- \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{\triangle_2}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} (\tau - t_j) - \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \sum_{j=1}^{k} v_j(\aleph(t_j)) \\
&+ \Big( \frac{\triangle_2 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} + \frac{t}{\triangle_3 + \triangle_4} \Big) g_2(\aleph) - \frac{g_1(\aleph)}{\triangle_1 + \triangle_2} \Big|.
\end{aligned} \tag{27}$$

After this, we have

$$\begin{aligned}
&\leq - \frac{1}{\Gamma(\xi)} \int_{t_k}^{t} (t-\zeta)^{\xi-1} |\psi(\zeta))| d\zeta \\
&- \frac{1}{(\triangle_1 + \triangle_2)} \Big( \frac{\triangle_1}{\Gamma(\xi)} \sum_{j=1}^{k} \int_{t_j - 1}^{t_j} (t_j - \zeta)^{\xi-1} |\psi(\zeta))| d\zeta \\
&+ \frac{\triangle_2}{\Gamma(\xi)} \int_{t_k - 1}^{t} (\tau - \zeta)^{\xi-1} |\psi(\zeta))| d\zeta \Big) \\
&- \Big( \sum_{j=1}^{k} (t - t_j) + \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{\triangle_2}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} (\tau - t_j) \\
&- \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \times \frac{1}{\Gamma(\xi - 1)} \sum_{j=1}^{k} \int_{t_j - 1}^{t_j} (t_j - \zeta)^{\xi-2} |\psi(\zeta))| d\zeta \\
&- \Big( \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \frac{1}{\Gamma(\xi - 1)} \\
&\times \int_{t_k}^{t} (\tau - \zeta)^{\xi-2} |\psi(\zeta))| d\zeta - \frac{\triangle_1}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} |v_i(\aleph(t_i))| - \Big( \sum_{j=1}^{k} (t - t_j) \\
&- \frac{\triangle_2 \triangle_4 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} - \frac{\triangle_2}{\triangle_1 + \triangle_2} \sum_{j=1}^{k} (\tau - t_j) - \frac{t \triangle_4}{\triangle_3 + \triangle_4} \Big) \sum_{j=1}^{k} |v_j(\aleph(t_j))| \\
&- \Big( \frac{\triangle_2 \tau}{(\triangle_1 + \triangle_2)(\triangle_3 + \triangle_4)} + \frac{t}{\triangle_3 + \triangle_4} \Big) |g_2(\aleph)| - \frac{|g_1(\aleph)|}{\triangle_1 + \triangle_2}.
\end{aligned} \tag{28}$$

Now using the defined conditions $u_1, u_2, u_3$ and $u_4$ to (28), then we get the following relation as

$$\begin{aligned}
&\leq \frac{\epsilon}{\Gamma(\xi+1)}(t-t_k)^{\xi} + \epsilon\Big[\frac{1}{(\triangle_1+\triangle_2)}\Big(\frac{\triangle_1}{\Gamma(\xi+1)}(t_j-t_{j-1})^{\xi} + \frac{\triangle_2}{\Gamma(\xi+1)}(\tau-t_{k-1})\Big]\\
&+\epsilon\Big(\sum_{j=1}^{k}(t-t_j) + \frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j)\\
&-\frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{\sum_{j=1}^{k}(t_j-t_{j-1})^{\xi-1}}{\Gamma(\xi)} + \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\\
&-\frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{\epsilon(\tau-t_k)^{\xi-1}}{\Gamma(\xi)} - \frac{\triangle_1 k\epsilon}{\triangle_1+\triangle_2} - \Big(\sum_{j=1}^{k}(t-t_j)\\
&-\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\sum_{j=1}^{k}(\tau-t_j) - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)k\epsilon\\
&-\Big(\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big)\chi_{g_2}|\aleph| + \mu_{g_2}\\
&-\frac{1}{\triangle_1+\triangle_2}(\chi_{g_1}|\aleph| + \mu_{g_1}).
\end{aligned} \tag{29}$$

Taking the supremum norm to (29), then we have

$$\leq \Omega\epsilon \tag{30}$$

where

$$\begin{aligned}
\Omega = &\Big[\frac{1}{\Gamma(\xi+1)} + \Big[\frac{1}{(\triangle_1+\triangle_2)}\Big(\frac{\triangle_1}{\Gamma(\xi+1)} + \frac{\triangle_2}{\Gamma(\xi+1)}\Big]\\
&+\Big(1 + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2}\\
&-\frac{\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\xi)} + \Big(\frac{\triangle_2\triangle_4\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_4}{\triangle_3+\triangle_4}\Big)\frac{1}{\Gamma(\xi)} - \frac{\triangle_1 k}{\triangle_1+\triangle_2}\\
&-\Big(1 - \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} - \frac{\triangle_2}{\triangle_1+\triangle_2} - \frac{t\triangle_4}{\triangle_3+\triangle_4}\Big)k\\
&-\Big(\frac{\triangle_2}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} + \frac{t}{\triangle_3+\triangle_4}\Big) - \frac{1}{\triangle_1+\triangle_2}\Big].
\end{aligned} \tag{31}$$

Now from (30), we have if we write $\aleph = T(\aleph)$ for the right hand side then,

$$|\aleph - T(\aleph)| \leq \epsilon\Omega \tag{32}$$

**Theorem 6.** *The solution of the propose problem is U-H stable and generalized U-H stable if $K < 1$.*

*Proof.* Let $\aleph \in \Psi$ be any solution of (1) and $\bar{\aleph} \in \Psi$ be unique solution.

$$|\aleph(t) - \bar{\aleph}(t)| = |\aleph(t) - T\bar{\aleph}(t)|,$$

$$= |\aleph(t) - T\aleph(t) + T\aleph(t) - T\bar{\aleph}(t)|, \tag{33}$$

$$\leq |\aleph(t) - T\aleph(t)| + |T\aleph(t) - T\bar{\aleph}(t)|, \tag{34}$$

Using (34),implies

$$|\aleph(t) - \bar{\aleph}(t)| \leq \epsilon\Omega + |\aleph(t) - T\bar{\aleph}(t)|.$$

Taking norm of both sides and the help of theorem (5) ,we have

$$||\aleph(t) - \bar{\aleph}(t)|| \leq \epsilon\Omega + K_{\xi}||\aleph - \bar{\aleph}||$$

Where

$$\begin{aligned}\mathbf{K}_\xi &= \Big[\Big(\frac{\tau^\xi}{\Gamma(\xi+1)} + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big| \times \frac{l\tau^\xi}{\Gamma(\xi+1)} + \Big|\frac{\triangle_2}{\triangle_1+\triangle_2}\Big| \times \frac{\tau^\xi}{\Gamma(\xi+1)} \\ &+ \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{l\tau^{\xi+1}}{\Gamma(\xi)} \\ &+ \Big|\frac{\triangle_2\triangle_4}{(r\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big| \times \frac{\tau^\xi}{\Gamma(\xi)}\Big)K_{\aleph} + \Big|\frac{\triangle_1}{\triangle_1+\triangle_2}\Big|K_I^i \\ &+ \Big|l + \frac{\triangle_2\triangle_4}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)}\Big|K_J^i + \Big|\frac{\triangle_2\tau}{(\triangle_1+\triangle_2)(\triangle_3+\triangle_4)} \\ &+ \frac{1}{\triangle_3+\triangle_4}\Big|\omega_{g_2} + \Big|\frac{1}{\triangle_1+\triangle_2}\Big|\omega_{g_1}\Big].\end{aligned}$$

Which yields

$$\|\aleph - \bar{\aleph}\| \le \frac{\Omega}{1-K_\xi}\epsilon,$$

Hence solution is U-H stable Further if $\psi : (0,1) \to R^+, \exists\psi(0) = 0$ Hence yields

$$\|\aleph - \bar{\aleph}\| \le \frac{\Omega}{1-K_\xi}\psi(\epsilon),$$

Hence solution is generalized U-H stable.

## 5. TEST PROBLEM

We take the follwing test problem for different values such as $\gamma = \frac{3}{2}, \triangle_1 = \frac{1}{5}, \triangle_2 = \frac{1}{8}, \triangle_3 = \frac{1}{2}, \triangle_4 = \frac{1}{3}, \tau = 1, D_f = W_f = 0.4, W_1 = W_2 = \mu_{g_1} = \mu_{g_2} = 0.1, D_1 = D_2 = 0.4, \chi_{g_1} = \chi_{g_2} = 0.001$

**Example 1.**

$$\begin{cases} D^{\frac{3}{2}}\aleph(t) = \dfrac{\exp(-\pi t)}{100+t^2}[\sin|\aleph(t)| + \aleph(\frac{t}{2}) + \aleph|(t-\frac{1}{2})|], t \in [0,1], t \ne \frac{1}{3}, \\ \triangle\aleph(\frac{1}{3}) = I(\aleph(\frac{1}{3}) = \dfrac{\sqrt(|\aleph|)}{60+\sqrt(|\aleph|)}, \\ \triangle\aleph'(\frac{1}{3})| = J_1(\aleph(\frac{1}{3}) = \dfrac{|\aleph|^{\frac{1}{3}}}{60+|\aleph|^{\frac{1}{3}}}, \\ \frac{1}{5}\aleph(0) + \frac{1}{8}\aleph(1) = \dfrac{\exp(-|\aleph|)}{60+|\aleph|^{\frac{1}{3}}}, \\ \frac{1}{2}\aleph'(0) - \frac{1}{3}\aleph'(1) = \dfrac{\exp(-\sin|\aleph|)}{100}. \end{cases}$$

*With the help of theorem 4, we can show the operators $E_0, E_1, E_2, E_3$ and $E_4$ are bounded and cotinuous. For this we take*

$$H = \{\aleph \in \Psi : \textit{there exist } \lambda = 1, \textit{ such that } \aleph = \lambda\tau\aleph\}.$$

*We demonstrate that for the above values, H is bounded in $\Psi$. Let $\aleph \in H$, $\lambda \in [0,1]$ such that $\|\aleph\| = \lambda\|\tau\aleph\|$. From (14) and (23), it follows that*

$$\|\aleph\| \le |\lambda|\Big(\|E_0\aleph\| + \|E_1\aleph\| + \|E_2\aleph\| + \|E_3\aleph\| + \|E_4\aleph\|\Big).$$

*After simplifying we have the following relation as*

$$\|\aleph\| \le \Big(0.409 + 0.702\|\aleph\| + (1 + 0.29 + 0.969 + 0.11 + 0.47) \times 0.4(1 + \|\aleph\|)\Big).$$

$$\|\aleph\| \le 1.541 + 2.022\|\aleph\|.$$

*Clearly H is bounded. For the uniquness result we take the values $K_\aleph = \frac{1}{12}, K_1^i = \frac{1}{12}, K_j^i = \frac{1}{24}, \omega_{g_1} = \frac{1}{6}, \omega_{g_2} = \frac{1}{16}, l = 1$, then we have*

$$\mathbf{K} = (0.7522 + 0.4628 + 0.2893 + 1.3017 + 0.1733) \times (0.0833) + (0.6902),$$

*this gives*

$$\mathbf{K} = \frac{9383}{10000}.$$

*We can conclude that the problem has a unique solution since all of Theorem 5's requirements are met. Furthermore, we have* $\mathbf{K} < 1$, *so in view of Theorem*(6), *the solution is Ulam-Hyers(U-H) Stable.*

**Example 2.**

$$\begin{cases} D^{\frac{7}{2}}\aleph(t) = \frac{1}{40+t^3}[\tan|\aleph(t)| + \aleph(\frac{t}{2}) + \aleph|(t-\frac{1}{2})|], t \in [0,1], t \neq \frac{1}{4}, \\ \triangle\aleph(\frac{1}{4}) = I(\aleph(\frac{1}{3}) = \frac{\cos(|\aleph|)}{100+(|\aleph|)}, \\ \triangle\aleph'(\frac{1}{4})| = J_1(\aleph(\frac{1}{4}) = \frac{\sin(|\aleph|)}{100+(|\aleph|)}, \\ \frac{1}{3}\aleph(0) + \frac{1}{5}\aleph(1) = \frac{\exp(|-\aleph|)}{40}, \\ \frac{1}{6}\aleph'(0) - \frac{1}{8}\aleph'(1) = \frac{\exp(-|\aleph|)}{40}. \end{cases}$$

*We take the follwing different values such as* $\gamma = \frac{7}{2}, \triangle_1 = \frac{1}{3}, \triangle_2 = \frac{1}{5}, \triangle_3 = \frac{1}{6}, \triangle_4 = \frac{1}{8}, \tau = 1, D_f = W_f = 0.4, W_1 = W_2 = \mu_{g_1} = \mu_{g_2} = 0.01, D_1 = D_2 = 0.2, \chi_{g_1} = \chi_{g_2} = 0.02$
*With the help of theorem 4, we can show the operators* $E_0, E_1, E_2, E_3$ *and* $E_4$ *are bounded and cotinuous. For this we take*

$$H = \{\aleph \in \Psi : \textit{there exist } \lambda = 1, \textit{ such that } \aleph = \lambda\tau\aleph\}.$$

*We demonstrate that for the given values, H is bounded in* $\Psi$. *Let* $\aleph \in H,\ \lambda \in [0,1]$ *such that* $||\aleph|| = \lambda||\tau\aleph||$. *It follows from* (14) *and* (23) *that*

$$||\aleph|| \leq |\lambda|\Big(||E_0\aleph|| + ||E_1\aleph|| + ||E_2\aleph|| + ||E_3\aleph|| + ||E_4\aleph||\Big).$$

*After simplifying we have the following relation as*

$$||\aleph|| \leq \Big(0.952 + 0.504||\aleph|| + (1 + 0.0030 + 0.372 + 0.072 + 0.059) \times 0.2(1 + ||\aleph||)\Big).$$

$$||\aleph|| \leq 1.253 + 0.805||\aleph||.$$

*Clearly H is bounded. For the uniquness result we take the values* $K_\aleph = \frac{1}{10}, K_l^i = \frac{1}{15}, K_j^i = \frac{1}{20}, \omega_{g_1} = \frac{1}{30}, \omega_{g_2} = \frac{1}{12}, l = 1$, *then we have*

$$\mathbf{K} = (0.086 + 0.053 + 0.189 + 0.351 + 0.048) \times (0.1) + (0.904),$$

*this gives*

$$\mathbf{K} = \frac{976}{1000}.$$

*We can conclude that the problem has a unique solution since all of Theorem 5's requirements are met. Furthermore, we have* $\mathbf{K} < 1$, *so in view of Theorem 6, the solution is Ulam-Hyers(U-H) Stable.*

## 6. CONCLUSION

A detailed analysis has been established for a class of impulsive FODEs involving Robbin boundary conditions. For the required analysis, we have used topological degree of Mahwin. Sufficient conditions have been developed for the existence and uniqueness of solution to the proposed problem. Also, an example has been given to demonstrate our theoretical results.

### Conflicts of Interest

We have no complect of interest.

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