Journal of Mathematical Modeling and Fractional Calculus
https://acspublisher.com/journals/index.php/jmmfc
Received:30-01-2025, **Revised:** 14-03-2025
Accepted: 07-04-2025, **Published:** 30-06-2025
https://doi.org/10.48165/jmmfc.2025.2101, 2(1),1-18.

Integral Transform and Generalized M-Series Fractional Integral Operators

Santosh Verma[a], Ankita Dwivedi*[a]

[a] Department of Mathematics, Guru Ghasidas Vishwavidyalaya,A Central University,Bilaspur-495009, India.
Corresponding author Email: (ankitadwivedi910@gmail.com)

Abstract: This paper presents new theorems that build upon existing research by extending the application of the Mellin transform within the framework of fractional integral operators. Traditionally, the Mellin transform has been a powerful tool for analyzing asymptotic behavior, scaling properties, and integral representations of special functions. However, by incorporating fractional integral operators, its analytical flexibility is significantly enhanced, allowing for a more in-depth study of special function properties, particularly in fractional calculus. Furthermore, the inclusion of the generalized I-function and M-series broadens this mathematical framework by generalizing established results and encompassing a wider class of special functions. Through their evaluation alongside fractional integral operators, this study introduces new integral representations, special cases, and applications that have not been previously explored. This extension greatly increases the applicability of these mathematical tools in diverse fields such as mathematical physics, engineering, and applied analysis, where fractional calculus and integral transforms play a crucial role in solving complex differential equations and boundary value problems. Ultimately, these theorems not only refine existing mathematical structures but also create new opportunities for future research on the interplay between integral transforms, special functions, and fractional calculus, contributing to both theoretical progress and practical advancements.

Keywords: M-series, I-function, Mellin transform and Fractional Integral.

1. Introduction

The study of special functions is fundamental to various branches of applied mathematics, physics and engineering. In particular generalized series such as the M-series provide a powerful framework for solving complex problems involving differential equations, integral equations and fractional calculus. The M-series denoted as ${}_{p}^{\varphi}M_{q}^{\zeta}$, extends classical hypergeometric functions and offers a more comprehensive approach to generalized operational techniques and transforms [26], [4].

In this paper, we explore the significant applications of the M-series in the context of integral transforms and generalized fractional integral operators. The M-series as expressed in the following form serves as a sophisticated generalization that can be adapted to various functional transformations and analytic techniques [1, 8]:

$$ {}_{p}^{\varphi}M_{q}^{\zeta}(d_1,\ldots,d_p;e_1,\ldots,e_q;z+2)=\sum_{k=0}^{\infty}\frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(z+2)^k}{\sqrt{\varphi k+\zeta}}. \tag{1} $$

In this representation, the parameters $d_1,\ldots,d_p$ and $e_1,\ldots,e_q$ govern the series while φ and ζ introduce additional flexibility into the formulation. This generalized M-series encompasses various classical functions as special cases and provides analytical tools for solving intricate problems in mathematical physics [2, 16].

The purpose of this research is to investigate the utility of the M-series in association with integral transforms such as the Mellin transform and generalized fractional integral operators. These mathematical tools are crucial for addressing a wide range of applications including physical systems, signal processing and fluid dynamics. By leveraging the properties of the M-series, we aim to explore new insights into how these operators can be extended and applied to more complex systems yielding novel solutions in both theoretical and applied contexts [15, 9, 7].

Our study enhances the applicability of fractional integral operators by incorporating the generalized I-function and the M-series enabling the analysis of a broader class of special functions. This development not only extends existing results but also introduces novel integral representations and previously unexplored special cases. Compared to traditional operators. Our approach improves computational efficiency by utilizing the Mellin transforms scaling properties streamlining complex integral evaluations [17, 19]. Furthermore, our findings highlight that the proposed operators have a broader application scope particularly in solving differential equations and boundary value problems across mathematical physics, engineering and applied analysis [25, 21].

2. Meijer G-Function as a Generalized M-Series

The generalized Meijer G-function can be expressed in the form of a hypergeometric function extending the concept of the M-series.

2.1. Meijer G-function

$$G^{\varphi,\zeta}_{p,q}\left[\begin{array}{c|c} d_p & \\ & (z+2) \\ e_q & \end{array}\right] = \frac{1}{2\pi i}\oint_L \frac{\sqrt{(e_{1,m}-s)}\sqrt{(1-d_{1,n}+s)}}{\sqrt{(d_{n+1,p}-s)}\sqrt{(1-e_{m+1,q}+s)}}(z+2)^s ds \tag{2}$$

where $\sqrt{.}$ is well known gamma function and $0 \le \varphi \le q$, $0 \le \zeta \le p$, $z \neq 0$. The definitions of Meijer G-function in the form of Hyper-geometric function [23], [5].

$$\begin{aligned} G^{\varphi,\zeta}_{p,q}\left[\begin{array}{c|c} d_p & \\ & (z+2) \\ e_q & \end{array}\right] &= \sum_{i=1}^{\varphi} \frac{\prod_{h=1}^{\varphi}\sqrt{e_h-e_i}\prod_{h=1}^{\zeta}\sqrt{1-d_h+e_i}}{\prod_{h=\varphi+1}^{q}\sqrt{1-(e_h-e_i)}\prod_{h=\zeta+1}^{p}\sqrt{d_h+e_i}}(z+2)^{e_i} \\ &\times {}_pF_{q-1}\left[\begin{array}{c|c} 1-d_p+e_q & \\ & (-1)^{p-\varphi-\zeta}(z+2) \\ 1-e_q+e_i & \end{array}\right], \end{aligned} \tag{3}$$

for p, q or $p=q$, $|z+2|<1$ and

$$\begin{aligned} G^{\varphi,\zeta}_{p,q}\left[\begin{array}{c|c} d_p & \\ & (z+2) \\ e_q & \end{array}\right] &= \sum_{i=1}^{\varphi} \frac{\prod_{h=1}^{\zeta}\sqrt{d_i-d_h}\prod_{h=1}^{\varphi}\sqrt{1-d_i+e_h}}{\prod_{h=\zeta+1}^{q}\sqrt{1-(d_i-d_h)}\prod_{h=\varphi+1}^{p}\sqrt{d_i+e_h}}(z+2)^{e_i-1} \\ &\times {}_pF_{q-1}\left[\begin{array}{c|c} 1-d_p+e_q & \\ & (-1)^{p-\varphi-\zeta}(z+2)^{-1} \\ 1-e_q+e_i & \end{array}\right], \end{aligned} \tag{4}$$

for p, q or $p=q$, $|z+2|>1$.

Special cases of the generalized M-series are mentioned in the following from (3):
(i) for $\varphi = \zeta = 1$, the generalized M-series is the generalized Meijer G-function in the form of hypergeometric from (3)

$$
\begin{aligned}
{}_pF_{q-1}\left[\begin{array}{c|c} 1-d_p+e_p & \\ & (-1)^{p-\varphi-\zeta}(z+2) \\ 1-e_p+e_i & \end{array}\right] &= {}^{\varphi}_{p}M^{\zeta}_{q}\left(d_1,d_2,\ldots,d_p \mid e_1,e_2,\ldots,e_q \mid (z+2)\right) \\
&= \sum_{k=0}^{\infty} \frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(z+2)^k}{k!}
\end{aligned} \tag{5}
$$

(a) After putting $\varphi = \zeta = 1$

$$
\begin{aligned}
{}_pF_{q-1}\left[\begin{array}{c|c} 1-d_p+e_p & \\ & (-1)^{p-1-1}(z+2) \\ 1-e_p+e_i & \end{array}\right] &= {}^{1}_{p}M^{1}_{q}\left(d_1,d_2,\ldots,d_p \mid e_1,e_2,\ldots,e_q \mid (z+2)\right) \\
&= \sum_{k=0}^{\infty} \frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(z+2)^k}{k!}
\end{aligned} \tag{6}
$$

(b) When $p = q = 0$, $\zeta = 1$ we have

$$
E_{\varphi}(z+2) = {}^{\varphi}_{0}M^{1}_{0}(-;-;(z+2)) = \sum_{k=0}^{\infty} \frac{(z+2)^k}{\sqrt{\varphi k+1}}, (\varphi > 0) \tag{7}
$$

where the symbol $E_{\varphi}(z+2)$ denotes the Mittag-Leffler function.

(c) Again, for $p = q = 0$, we have

$$
E_{\varphi,\zeta}(z+2) = {}^{\varphi}_{0}M^{\zeta}_{0}(-;-;(z+2)) = \sum_{k=0}^{\infty} \frac{(z+2)^k}{\sqrt{\varphi k+\zeta}}, (\varphi > 0, \zeta > 0) \tag{8}
$$

where the symbol $E_{\varphi,\zeta}(z+2)$ denotes the two-index Mittag-Leffler function.
(d) If we set $p = q = 1$, $\alpha = \sigma \in \mathbb{C}$, and $\beta_1 = 1$, then the generalized M-series reduces to the generalized Mittag-Leffler function [2][25].

$$
E_{\varphi,\zeta}(z+2) = {}^{\varphi}_{1}M^{\zeta}_{1}(-;-;(z+2)) = \sum_{k=0}^{\infty} \frac{(\sigma)_k}{(1)_k}\frac{(z+2)^k}{\sqrt{\varphi k+\zeta}} = \sum_{k=0}^{\infty} \frac{(\sigma)_k}{\sqrt{\varphi k+\zeta}}\frac{(z+2)^k}{k!}, (\varphi > 0, \zeta > 0) \tag{9}
$$

3. Special cases of the generalized M-series

Special cases of the generalized M-series are mentioned in the following [12], [6]
Case (i): for $\varphi = \zeta = 1$, the generalized M-series is the generalized Meijer G-function in the form of hypergeometric form from (4)[3], [10], [18].

$$
\begin{aligned}
{}_qF_{p-1}\left[\begin{array}{c|c} 1-d_i+e_q & \\ & (-1)^{q-\varphi-\zeta}(z+2)^{-1} \\ 1-d_i+d_p & \end{array}\right] &= {}^{\varphi}_{p}M^{\zeta}_{q}\left(d_1,d_2,\ldots,d_p \mid e_1,e_2,\ldots,e_q \mid (z+2)\right) \\
&= \sum_{k=0}^{\infty} \frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(z+2)^k}{k!}
\end{aligned} \tag{10}
$$

while putting $\varphi = \zeta = 1$ then we get,

$$
\begin{aligned}
{}_qF_{p-1}\left[\begin{array}{c|c} 1-d_i+e_q & \\ & (-1)^{q-1-1}(z+2)^{-1} \\ 1-d_i+d_p & \end{array}\right] &= {}^{1}_{p}M^{1}_{q}\left(d_1,d_2,\ldots,d_p \mid e_1,e_2,\ldots,e_q \mid (z+2)\right) \\
&= \sum_{k=0}^{\infty} \frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(z+2)^k}{k!}
\end{aligned} \tag{11}
$$

Case (ii): When we set $p = q = 0$, $\zeta = 1$, the resulting equation is equivalent to the previously referenced equation, presumably labeled as (7). which denotes the Mittag-Leffler function. **Case (iii):** When we set $p = q = 0$, the resulting equation is equivalent to the previously referenced equation, presumably labeled as (8). Which denotes the Mittag-Leffler function.
Case (iv): When we set $p = q = 0$, the resulting equation is equivalent to the previously referenced equation, presumably labeled as (9). which denotes the Mittag-Leffler function.

4. The Generalized Fractional Integral Operators

Now, we recall the definition of generalized fractional integral operators involving the I-function. The new generalized fractional integral operators, involving the I-function as the kernel [14], [13]

$$P_{0,w;r}^{\psi,\xi}[f(w)] = 2^h r w^{-\psi-r\xi-1} \int_0^w t^\psi (w^r - t^r)^\xi \times I_{p_i,q_i;r}^{\delta,\aleph}\left[\rho U \left| \frac{(a_j,\alpha_j)_{1,\aleph};(a_{ji},\alpha_{ji})_{\aleph+1;p_i}}{(b_j,\beta_j)_{1,\delta};(a_{ji},\beta_{ji})_{\delta+1;q_i}}\right.\right] f(t)dt \quad (12)$$

where $h > 0$,

$$U = \left(\frac{t^r}{w^r}\right)^\tau \left(1 - \frac{t^r}{w^r}\right)^v \quad (13)$$

and

$$V = \left(\frac{w^r}{t^r}\right)^\tau \left(1 - \frac{w^r}{t^r}\right)^v \quad (14)$$

The sufficient conditions for mathematical operators refers to the rules or criteria that,

$$\begin{cases} (i) 1 \le p,q < \infty, p^{-1} + q^{-1} = 1; \\ (ii) \mathscr{R}\left(2\mu + r\tau\left(\frac{b_j}{\beta_j}\right)\right) > -q^{-1}; \\ \mathscr{R}\left(2\xi + r\tau\left(\frac{b_j}{\beta_j}\right)\right) > -q^{-1}; \\ \mathscr{R}\left(2\omega + \xi + r\tau\left(\frac{b_j}{\beta_j}\right)\right) > -p^{-1}; j = 1,2\ldots,n \\ (iii) f(x) \in L_p(0,\infty); \\ (iv) |\arg\rho| \le \pi\frac{\Theta}{2}, \Theta > 0 \end{cases} \quad (15)$$

$$\Theta = \sum_{j=1}^{\aleph}(\alpha_i) + \sum_{j=1}^{\delta}(\beta_i) - \max_{1\le i\le r}\left[\sum_{j=\aleph+1}^{p_i}(\alpha_i) + \sum_{j=\delta+1}^{q_i}(\beta_i)\right] \quad (16)$$

To obtain the simplified form of Eq. (12) it is essential to analyze the integrals components and the behavior of the function $I_{p_i,q_i;r}^{\delta,\aleph}$ behaves. The simplified form of the equation is as follows:

$$P_{0,w;r}^{\psi,\xi}[f(w)] = 2^h r w^{-\psi-r\xi-1} \int_0^w t^\psi (w^r - t^r)^\xi I_{p_i,q_i;r}^{\delta,\aleph}\left[\rho \frac{t^r}{w^r}\left(1 - \frac{t^r}{w^r}\right) \left| \frac{(a_j,\alpha_j)}{(b_j,\beta_j)}\right.\right] f(t)\,dt \quad (17)$$

Rearranging the powers and simplifying the above equation

$$w^{-\psi-r\xi-1} t^\psi (w^r - t^r)^\xi = w^{-\psi-r\xi-1} t^\psi \left[w^r\left(1 - \left(\frac{t}{w}\right)^r\right)\right]^\xi \quad (18)$$

Further this simplifies to:

$$P_{0,w;r}^{\psi,\xi}[f(w)] = 2^h r w^{-\psi-r\xi-1} \int_0^w t^\psi w^{r\xi}\left(1 - \frac{t^r}{w^r}\right)^\xi I_{p_i,q_i;r}^{\delta,\aleph}\left[\rho \frac{t^r}{w^r}\left(1 - \frac{t^r}{w^r}\right) \left| \frac{(a_j,\alpha_j)}{(b_j,\beta_j)}\right.\right] f(t)\,dt \quad (19)$$

Factor out terms independent of t:

$$P_{0,w;r}^{\psi,\xi}[f(w)] = 2^h r w^{-\psi-1} \int_0^w t^\psi \left(1 - \frac{t^r}{w^r}\right)^\xi I_{p_i,q_i;r}^{\delta,\aleph}\left[\rho \frac{t^r}{w^r}\left(1 - \frac{t^r}{w^r}\right) \left| \frac{(a_j,\alpha_j)}{(b_j,\beta_j)}\right.\right] f(t)\,dt \quad (20)$$

4.1. Expansion of the Hypergeometric-Like Function

Assume the generalized hypergeometric function $I^{\delta,\aleph}_{p_i,q_i;r}$ can be expanded as a series:

$$I^{\delta,\aleph}_{p_i,q_i;r}\left[\rho \frac{t^r}{w^r}\left(1-\frac{t^r}{w^r}\right)\Big|\ldots\right] \approx \sum_{n=0}^{\infty} c_n \left(\frac{t^r}{w^r}\left(1-\frac{t^r}{w^r}\right)\right)^n \tag{21}$$

Substitute this into the integral:

$$P^{\psi,\xi}_{0,w;r}[f(w)] \approx 2^h r w^{-\psi-1} \int_0^w t^\psi \left(1-\frac{t^r}{w^r}\right)^\xi \sum_{n=0}^{\infty} c_n \left(\frac{t^r}{w^r}\left(1-\frac{t^r}{w^r}\right)\right)^n f(t)\,dt \tag{22}$$

We can simplify the series expansion (leading order term $n=0$)

$$P^{\psi,\xi}_{0,w;r}[f(w)] \approx 2^h r w^{-\psi-1} \int_0^w t^\psi \left(1-\frac{t^r}{w^r}\right)^\xi f(t)\,dt \tag{23}$$

To expand and solve the integral problem in Eq. (23), we proceed as follows: We need to analyze each part of the integral and determine it, whether by using numerical techniques.
Consider Special Cases for Simplification To simplify the integral, assume $f(t)=1$, so we get:

$$P^{\psi,\xi}_{0,w;r}[1] \approx 2^h r w^{-\psi-1} \int_0^w t^\psi \left(1-\frac{t^r}{w^r}\right)^\xi dt \tag{24}$$

By Solving the Integral, the integral becomes:

$$I(w) = \int_0^w t^\psi \left(1-\frac{t^r}{w^r}\right)^\xi dt \tag{25}$$

This is a non-trivial integral that might require numerical integration or approximations like series expansions for small or large t/w.

Approximations for Specific Cases which we use:
If $t \ll w$, the term $\left(1-\frac{t^r}{w^r}\right)^\xi$ can be expanded using a binomial expansion:

$$\left(1-\frac{t^r}{w^r}\right)^\xi \approx 1-\xi\frac{t^r}{w^r}+O\left(\left(\frac{t^r}{w^r}\right)^2\right) \tag{26}$$

While Substituting this expansion into the integral gives:

$$I(w) \approx \int_0^w t^\psi \left(1-\xi\frac{t^r}{w^r}\right) dt = \int_0^w t^\psi\,dt - \xi\frac{1}{w^r}\int_0^w t^{\psi+r}\,dt \tag{27}$$

Standard power-law integrals involve integrating expressions:

$$\int_0^w t^\psi\,dt = \frac{w^{\psi+1}}{\psi+1}, \quad \int_0^w t^{\psi+r}\,dt = \frac{w^{\psi+r+1}}{\psi+r+1} \tag{28}$$

while solving the above equations, the approximation becomes:

$$I(w) \approx w^{\psi+1}\left(\frac{1}{\psi+1}-\xi\frac{1}{\psi+r+1}\right) \tag{29}$$

4.2. *Example Using the Beta Function and $f(w) = w^2$*

Let's assume $I^{\delta,\aleph}_{p_i,q_i;r}$ reduces to the $B(a,b)$ Suppose we take $f(w) = w^2$ is a simple quadratic function, $\psi = 1$, $\zeta = 1$, $r = 2$, $h = 0$ while substituting in Eq. (12)

$$P^{1,1}_{0,w;2}[w^2] = 2^0 2w^{-1-2-1} \int_0^w t^1(w^2 - t^2).B(a,b)t^2 dt. \tag{30}$$

The Beta function $B(a,b)$ is independent of t, so it can be factored out. Now while simplify we get:

$$P^{1,1}_{0,w;2}[w^2] = 2B(a,b)w^{-4} \int_0^w t^3(w^2 - t^2)dt.$$

by solving integral part we get:

$$\int_0^w t^3(w^2 - t^2)dt = \frac{w^6}{12}$$

Now come to final expression, where $a = 8$, $b = 9$

$$P^{1,1}_{0,w;2}[w^2] = 2B(a,b)\frac{w^6}{12} \Rightarrow \frac{B(a,b)}{6}.w^2 \tag{31}$$

while using the Eq. (31) we plot the graph

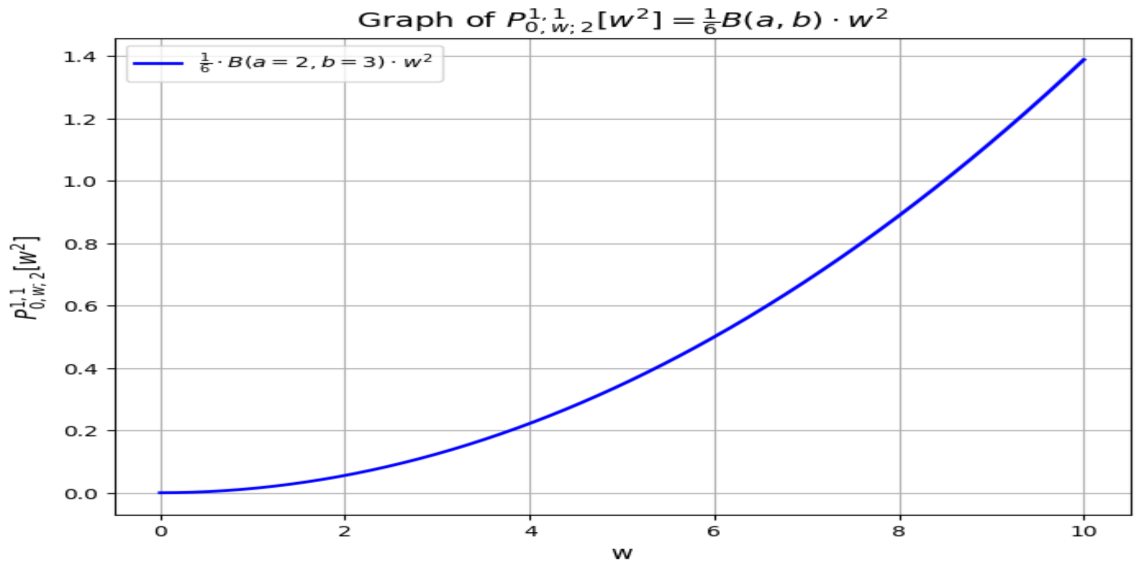

Figure 1: The function starts near zero at $w = 0$ and follows a quadratic growth reaching approximately 0.00016 at $w = 10$

Figure 1, the graph exhibits a smooth upward curvature characteristic of polynomial growth illustrating the interplay between special functions and algebraic structures. Physically, such functions are fundamental in statistical mechanics, probability distributions and combinatorial analysis where they influence normalization factors and scaling behavior. This visualization highlights the Beta functions role in shaping polynomial dependencies offering valuable insights into applied mathematical modeling.

$$P^{\phi,\xi}_{w,\infty;r}[f(w)] = 2^h r w^\omega \int_w^\infty t^{-\omega - r\xi - 1}(t^r - w^r)^\xi \times I^{\delta,\aleph}_{p_i,q_i;r}\left[\rho V \middle| \begin{matrix} (a_j,\alpha_j)_{1,\aleph}; (a_{ji},\alpha_{ji})_{\aleph+1;p_i} \\ (b_j,\beta_j)_{1,\delta}; (a_{ji},\beta_{ji})_{\delta+1;q_i} \end{matrix}\right] f(t)dt \tag{32}$$

Substituting this Eq. (32) expression for $P^{\psi,\xi}_{0,w;r}[f(w)]$, we get approximate solution: The equation is written as:

$$P^{\phi,\zeta}_{w,\infty;r}[1] \approx 2^h r w^\omega \int_w^\infty t^{-w-1}\left(1 - \frac{w^r}{t^r}\right)^\xi dt$$

Where $(P^{\phi,\zeta}_{w,\infty;r}[1])$ is some form of generalized function, likely dependent on parameters (ϕ) and (ζ). $(2^h r w^{\omega})$ is an algebraic prefactor. Expression of the integral .

$$\int_w^\infty t^{-w-1}\left(1-\frac{w^r}{t^r}\right)^{\xi} dt$$

This looks complex, but simplify it by applying the Substitution method:

If the variables (w) or (t) have specific known values, try substituting them in. Asymptotic

$$P^{\psi,\xi}_{0,w;r}[f(w)] \approx 2^h r w^{-\psi-1}\left(\frac{w^{\psi+1}}{\psi+1}-\xi\frac{w^{\psi+1}}{\psi+r+1}\right)$$

Simplifying:

$$P^{\psi,\xi}_{0,w;r}[f(w)] \approx 2^h r\left(\frac{1}{\psi+1}-\xi\frac{1}{\psi+r+1}\right)$$

Assuming approximation $(f(t)=1)$.

$$P^{\psi,\xi}_{0,w;r}[1] \approx 2^h r\left(\frac{1}{\psi+1}-\xi\frac{1}{\psi+r+1}\right)$$

$$I(w) \approx w^{\psi+1}\left(\frac{1}{\psi+1}-\xi\frac{1}{\psi+r+1}\right)$$

Now while solving the equation above, we get: To solve the given integral:

$$I(w)=\int_w^\infty t^{-w-1}\left(1-\frac{w^r}{t^r}\right)^{\xi} dt$$

5. Analysis and Evaluation of the Integral

In this section, we focus on the integral part and solve it for specific or general values of the parameters (w,r,ξ). The integral under consideration is of the form:

$$I(w)=\int_w^\infty t^{-w-1}\left(1-\frac{w^r}{t^r}\right)^{\xi} dt \tag{33}$$

This integral has a power-law term (t^{-w-1}) and a more complex term $(\left(1-\frac{w^r}{t^r}\right)^{\xi})$. Then change of variable

Let's a substitution to simplify the powers of (t). Let we see:

$$u=\frac{t}{w} \quad \Rightarrow \quad t=uw \quad \Rightarrow \quad dt=w\,du \tag{34}$$

Thus, the integral becomes:

$$I(w)=\int_1^\infty (uw)^{-w-1}\left(1-\frac{1}{u^r}\right)^{\xi} w\,du \tag{35}$$

Simplifying the powers of (w) and (u):

$$I(w)=w^{-w}\int_1^\infty u^{-w-1}\left(1-\frac{1}{u^r}\right)^{\xi} du \tag{36}$$

Special cases and approximations

Case 1: Small (r) and $(\xi = 1)$
When $(\xi = 1)$, the integral simplifies to:

$$I(w) = w^{-w} \int_1^{\infty} u^{-w-1} \left(1 - \frac{1}{u^r}\right) du \tag{37}$$

This can be split into two parts:

$$I(w) = w^{-w} \left(\int_1^{\infty} u^{-w-1}\, du - \frac{1}{w^r} \int_1^{\infty} u^{-w-r-1}\, du\right) \tag{38}$$

The integrals are standard and evaluate as follows:

$$\int_1^{\infty} u^{-w-1}\, du = \frac{1}{w}$$

$$\int_1^{\infty} u^{-w-r-1}\, du = \frac{1}{w+r}$$

Thus, the integral becomes:

$$I(w) = w^{-w} \left(\frac{1}{w} - \frac{1}{w^r(w+r)}\right) \tag{39}$$

Case 2 : General (ξ)
For a general (ξ), this integral is more difficult to solve analytically. However, it can be approximated for large (t) (or equivalently large (u)). For large (u):

$$\left(1 - \frac{1}{u^r}\right)^{\xi} \approx 1 - \frac{\xi}{u^r} \tag{40}$$

This approximation leads to:

$$I(w) \approx w^{-w} \int_1^{\infty} u^{-w-1} \left(1 - \frac{\xi}{u^r}\right) du \tag{41}$$

This gives a similar structure to Case 1, with corrections depending on (ξ).
Final expression, after simplification, for $(\xi = 1)$, the result is approximately:

$$I(w) \approx w^{-w} \left(\frac{1}{w} - \frac{1}{w^r(w+r)}\right) \tag{42}$$

For general (ξ), the integral may not have a simple closed form, but can be approximated numerically or using series expansions.

5.1. Example with Beta function and $f(w) = w^2$

Putting the value $f(w) = w^2, \omega = 1, \phi = 2, \zeta = 2, r = 2, h = o$ in Eq. (32) after solving this we get the equation:

$$P^{2,2}_{0,w;2}[w^2] = -\frac{13}{15w} B(a,b) \tag{43}$$

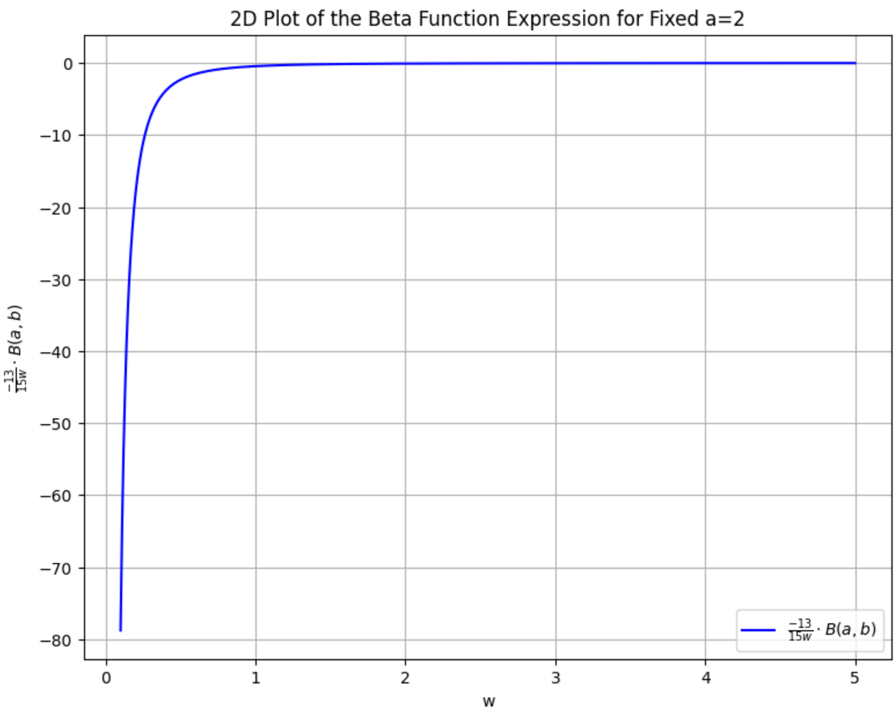

Figure 2: The function approaches stability near zero for $w > 1$ exhibiting minimal variation beyond this point.

Figure 2, illustrates a Beta function expression for $a = 2$. The curve exhibits a sharp decline for small w before stabilizing near zero. This behavior is crucial in probability distributions, Bayesian statistics, and statistical physics. The asymptotic nature of the function makes it relevant for modeling decay processes and normalization in applied mathematics. This visualization highlights the Beta function's transformation properties and its analytical significance.
The I-function, which is a more generalized version of the Fox H-function, was introduced by Saxena et al. [21]. It is defined using a contour integral of the Mellin-Barnes type. This type of integral is used in complex analysis and involves integrating along a specific contour in the complex plane:[24], [20], [22]

$$I^{\delta,\aleph}_{p_i,q_i;r}\left[(z+2)\left|\frac{(a_j,\alpha_j)_{1,\aleph};(a_{ji},\alpha_{ji})_{\aleph+1;p_i}}{(b_j,\beta_j)_{1,\delta};(a_{ji},\beta_{ji})_{\delta+1;q_i}}\right.\right]=\frac{1}{2\pi\wp}\int_L \mathscr{E}(\Theta)(z+1)^{\mathscr{H}}\,d\mathscr{H} \tag{44}$$

when$f(x)$ given is

$$\mathscr{E}(\Theta)=\frac{\prod_{j=1}^{\delta}\sqrt{(b_j-\beta_j\Theta)}\prod_{j=1}^{\aleph}\sqrt{(1-a_j+\alpha_j\Theta)}}{\sum_{i=1}^{r}\prod_{j=\delta+1}^{q_i}\sqrt{(1-b_{ji}+\beta_{ji}\Theta)}\prod_{j=\aleph+1}^{p_i}\sqrt{(a_{ji}-\alpha_{ji}\Theta)}} \tag{45}$$

where value of 1 and δ is
p_i, q_i $(i = 1,2,...,r)$ $\delta, \aleph$ are the integers satisfying $0 \le \delta \le p_i, 0 \le \aleph \le q_i, \alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$are the real and positive number, and a_j, b_j, a_{ji}, b_{ji} are the complex number. L is a suitable contour of the Mellin-Bernes type running from $\gamma - i\,\Omega\ to\ \gamma + i\Omega$ (Ω is real in the complex $\exists$-plane)
for $r = 1$ reduces to for to Fox $\mathscr{H}$-function

$$I^{\delta,\aleph}_{p_i,q_i;r}\left[\begin{array}{c|cc}(z+2) & (a_j,\alpha_j)_{1,\aleph}; & (a_{ji},\alpha_{ji})_{\aleph+1;p_i}\\ & (b_j,\beta_j)_{1,\delta}; & (a_{ji},\beta_{ji})_{\delta+1;q_i}\end{array}\right]=\mathscr{H}^{\delta,\aleph}_{p_i,q_i;r}\left[\begin{array}{c|cc}(z+2) & (a_j,\alpha_j)_{1,\aleph}; & (a_{ji},\alpha_{ji})_{\aleph+1;p_i}\\ & (b_j,\beta_j)_{1,\delta}; & (a_{ji},\beta_{ji})_{\delta+1;q_i}\end{array}\right] \tag{46}$$

6. Representation of the Generalized M-Series within the Framework of Generalized Fractional Integral Operators

In this section, we derived the image formula for the generalized M-series using the generalized fractional integral operators, expressed in terms of the I-function as the kernel.

Theorem 6.1: Let $a > 0, x > 0, \varphi, \zeta, \eta, \Delta \in \mathbb{C}, \mathscr{R}(\zeta) > 0, \mathscr{R} > 0, \mathscr{R}(\Delta) > 0, 1 \leq p \leq 2$, then the fractional integration $P_{0,w;r}^{\psi,\xi}$ of the product of M-L function exists under the condition $p^{-1} + q^{-1}$

$$\mathscr{R}\left(\mu + r\tau\left(\frac{b_j}{\beta_j}\right)\right) > -q^{-1}; \mathscr{R}\left(\xi + r\tau\left(\frac{b_j}{\beta_j}\right)\right) > -q^{-1} \tag{47}$$

then there holds the following formula,

$$P_{0,w;r}^{\psi,\xi}\left((t^{\eta-1})\,{}_p^{\varphi}M_q^{\zeta}\left(ax^{\Delta}+1\right)\right)(x) = x^{-\eta}\sum_{k=0}^{\infty}\frac{(d_1)_k\ldots(d_p)_k}{(e_1)^k\ldots(e_q)^k}\frac{(ax^{\Delta}+1)^k}{\Gamma(\varphi k+\zeta)}$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{48}$$

Proof: We assume Λ to be the left-hand side of the above *equation*. Using the definition of the generalized M-series and the generalized fractional integral operators Eq. (15) on the left-hand side of the *above equation*, we have:

$$\Lambda = rx^{\varphi-r\zeta-1}\int_0^x t^{\mu+\eta-1}(x^r-t^r)^{\xi}\left\{\frac{1}{2\pi\wp}\int_L \varphi(\zeta)(\rho U)^{\zeta}d\zeta\right\}\times\sum_{k=0}^{\infty}\frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(ax^{\Delta}+1)^k}{\Gamma(\varphi k+\zeta)} \tag{49}$$

Now by changing the order of integration, which is valid under the given theorem, we get:

$$\Lambda = rx^{-\varphi-r\zeta-1}\sum_{k=0}^{\infty}\frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{a^k}{\Gamma(\varphi k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}x^{r\varphi-r\tau\eta}\varphi(\zeta)$$
$$\left\{\int_0^x t^{\mu+\eta+\Delta k+r\tau\eta-1}\left(1-\frac{t^r}{x^r}\right)^{\xi+v\zeta}dt\right\} \tag{50}$$

Let the substitution $\frac{t^r}{x^r} = \mathbf{B}$, and then $t = x\mathbf{B}^{\frac{1}{r}}$ in Eq. (50), we get:

$$\Lambda = x^{\eta-1}\sum_{k=0}^{\infty}\frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(ax^{\Delta}+1)^k}{\Gamma(\varphi k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}x^{r\varphi-r\tau\eta}\varphi(\zeta)\left\{\int_0^1 \mathbf{B}^{\frac{\mu+\eta+\Delta k}{r}+\tau\zeta-1}(1-\mathbf{B})^{\xi+v\zeta}d\mathbf{B}\right\}d\zeta \tag{51}$$

Using the definition of the well-known beta function in the inner integral, we have:

$$\Lambda = x^{\eta-1}\sum_{k=0}^{\infty}\frac{(d_1)_k\cdots(d_p)_k}{(e_1)_k\cdots(e_q)_k}\frac{(ax^{\Delta}+1)^k}{\Gamma(k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}\varphi(\zeta)\frac{\sqrt{\frac{\mu+\eta+\Delta k}{r}+\tau\zeta}\sqrt{1+\xi+v\zeta}}{1+\xi+\mu+\eta+\frac{\Delta k}{r}+(\tau+v\zeta)}d\zeta \tag{52}$$

Theorem 6.2: Let $a > 0, x > 0, \varphi, \zeta, \eta, \Delta \in \mathbb{C}, \mathscr{R}(\zeta) > 0, \mathscr{R} > 0, \mathscr{R}(\Delta) > 0, 1 \leq p \leq 2$, then the fractional integration $P_{0,w;r}^{\psi,\xi}$ of the product of M-L function exists under the condition $p^{-1} + q^{-1}$ then there holds the following formula,

$$P_{x,\infty;r}^{\phi,\xi}\left((t^{\eta-1})\,{}_p^{\varphi}M_q^{\zeta}\left(\frac{a}{t^{\Delta}}+1\right)\right)(x) = x^{-\eta}\sum_{k=0}^{\infty}\frac{(d_1)_k\ldots(d_p)_k}{(e_1)^k\ldots(e_q)^k}\frac{(\frac{a}{t^{\Delta}}+1)^k}{\Gamma(\varphi k+\zeta)}$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{53}$$

Proof: We assume Λ be one the left hand side above *equation* using the definition of generalized M-series and the generalized fractional integral operators on the left-hand side of *above equation*, we have

$$\Lambda_2 = rx^{\phi}\int_0^x t^{-\phi-\eta-r\xi-1}(x^r-t^r)^{\xi}\left\{\frac{1}{2\pi\wp}\int_L \varphi(\zeta)(\rho V)^{\zeta}d\zeta\right\}\times\sum_{k=0}^{\infty}\frac{(d_1)_k...(d_p)_k}{(e_1)_k...(e_q)_k}\frac{((\frac{a}{t^{\Delta}}+1)^k}{\Gamma(\varphi k+\zeta)} \tag{54}$$

By reversing the order of integration, which is permissible according to the given theorem, we get:

$$\Lambda_2 = rx^{\phi}\sum_{k=0}^{\infty}\frac{(d_1)_k...(d_p)_k}{(e_1)_k...(e_q)_k}\frac{(a)^k}{\Gamma(\varphi k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}x^{r\tau\eta}\varphi(\zeta)\left\{\int_0^x t^{\mu+\eta+\Delta k+r\tau\eta-1}\left(1-\frac{x^r}{t^r}\right)^{\xi+v\zeta}dt\right\} \tag{55}$$

Let the substitution $\frac{x^r}{t^r}=\mathbf{W}$ and then $t=\frac{x}{\mathbf{W}^{\frac{1}{r}}}$ in Eq. (55) we get:

$$\Lambda_2 = x^{-\eta-1}\sum_{k=0}^{\infty}\frac{(d_1)_k...(d_p)_k}{(e_1)_k...(e_q)_k}\frac{((ax^{-\Delta})^k}{\Gamma(\varphi k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}\varphi(\zeta)\left\{\int_0^1 \mathbf{B}^{(\phi+\zeta+\frac{\phi k}{r})+\tau\zeta-1}(1-\mathbf{B})^{\xi+v\zeta}d\mathbf{B}\right\}d\zeta \tag{56}$$

Using the defination of the well-known beta function in the inner integral, we have:

$$\Lambda_2 = x^{-\eta+r\varphi-1}\sum_{k=0}^{\infty}\frac{(d_1)_k...(d_p)_k}{(e_1)_k...(e_q)_k}\frac{((ax^{-\Delta}+1)^k}{\Gamma(\varphi k+\zeta)}\frac{1}{2\pi\wp}\int_L \rho^{\zeta}\varphi(\zeta)\frac{\sqrt{\phi+\eta+\Delta k/r+\tau\zeta}\sqrt{1+\xi+v\zeta}}{1+\xi+\phi+\eta+\Delta k/r+(\tau+v\zeta)}d\zeta \tag{57}$$

Interpreting the right hand side Eq. (57).

6.1. Special Cases:

Corollary 1. If we put $\varphi=1,\zeta=2$ and $p=q=2$ (Theorem 6.1, Theorem 6.2). We Obtain following interesting results on the right, and it is known as the generalized Hypergeometric function (G-Meijer).

$$\begin{aligned}
&P^{\psi,\xi}_{0,w;r}\left(t^{\eta-1}\ {}^1_2M^2_2(d_1\cdots d_pe_1\cdots e_q;ax^{\Delta+1})\right)(x)\\
&\quad= x^{\eta-1}{}_pF_q\left[\begin{matrix}d_1,\ldots,d_p & ; & ax^{\xi}+1\\ e_1,\ldots,e_q & & \end{matrix}\right]\\
&\quad\times I^{\delta,\aleph+2}_{p+2,q+1;r}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right].
\end{aligned} \tag{58}$$

Corollary 2. If we put $\varphi=1,\zeta=2$ and $p=q=2$ (Theorem 6.2) then we get

$$\begin{aligned}
&P^{\phi,\xi}_{x,\infty;r}\left(t^{\eta-1}\ {}^1_2M^2_2(d_1\cdots d_pe_1\cdots e_q;\frac{a}{t^{\Delta}}+1)\right)(x)\\
&\quad= x^{\eta-1}{}_pF_q\left[\begin{matrix}d_1,\ldots,d_p & ; & ax^{\xi}+1\\ e_1,\ldots,e_q & & \end{matrix}\right]\\
&\quad\times I^{\delta,\aleph+2}_{p+2,q+1;r}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right].
\end{aligned} \tag{59}$$

Corollary 3. If we put $\varphi = 1, \zeta = 2$ and $p = q = 0$ then we get:

$$P_{0,w;r}^{\psi,\xi}\left(t^{\eta-1}\,{}_0^1M_0^2(-;-;ax^{\Delta+1})\right)(x) = x^{\eta-1}E_{\varphi\zeta}(ax^{\Delta}+1)$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho \left| \begin{matrix} (a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2) \\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{60}$$

Corollary 4. If we put $\varphi = 1, \zeta = 2$ and $p = q = 0$ then we get:

$$P_{x,\infty;r}^{\phi,\xi}\left(t^{\eta-1}\,{}_0^1M_0^2\left(d_1\cdots d_p, e_1\cdots e_q;\frac{a}{t^{\Delta}}+1\right)\right)(x)$$
$$= x^{\eta-1}E_{\varphi\zeta}(ax^{\Delta}+1)$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho \left| \begin{matrix} (a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2) \\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{61}$$

Corollary 5. If we put $p = q = 0$ then we get:

$$P_{0,w;r}^{\psi,\xi}\left(t^{\eta-1}\,{}_0^{\varphi}M_0^{\zeta}(-;-;ax^{\Delta+1})\right)(x) = x^{\eta-1}E_{\varphi\zeta}(ax^{\Delta}+1)$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho \left| \begin{matrix} (a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2) \\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{62}$$

Corollary 6. If we put $p = q = 0$ then we get:

$$P_{x,\infty;r}^{\phi,\xi}\left(t^{\eta-1}\,{}_0^{\varphi}M_0^{\zeta}\left(-;-;\frac{a}{t^{\Delta}}+1\right)\right)(x)$$
$$= x^{\eta-1}E_{\varphi\zeta}(ax^{\Delta}+1)$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho \left| \begin{matrix} (a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2) \\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{63}$$

Corollary 7. If we put $p = q = 1, d_1 = \Upsilon \in \mathbb{C}$ and $e_1 = 1$ then we get:

$$P_{0,w;r}^{\psi,\xi}\left(t^{\eta-1}\,{}_1^{\Upsilon}M_1^1(\Upsilon;1;ax^{\Delta+1})\right)(x) = x^{\eta-1}E_{\varphi\zeta}(ax^{\Delta}+1)$$
$$\times I_{p+2,q+1;r}^{\delta,\aleph+2}\left[\rho \left| \begin{matrix} (a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2) \\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]. \tag{64}$$

Corollary 8. If we put $p=q=1, d_1=\Upsilon\in\mathbb{C}$ and $e_1=1$ then we get:

$$\begin{aligned}
&P^{\phi,\xi}_{x,\infty;r}\left(t^{\eta-1}\,{}^{\varphi}_{1}M^{\zeta}_{1}\left(\Upsilon;1;\frac{a}{t^{\Delta}}+1\right)\right)(x)\\
&=x^{-\eta}E_{\varphi\zeta}(ax^{\Delta}+1)\\
&\times I^{\delta,\aleph+2}_{p+2,q+1;r}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+vv^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right].
\end{aligned}\tag{65}$$

7. Certain integral transforms offer a valuable method for simplifying complex functions

Here in this section we will provide some very important outcomes of several theorems connected with the transform of Mellin [11].

Defination: The Mellin transform of the is define

$$\mathscr{M}[f;s]=f(s)=\int_0^{\infty}f(t)t^{s-1}dt;\qquad \boxed{s=y+jz}\;;\;\boxed{y<y_1<y_2}\tag{66}$$

wherey_1 and y_2 depend on the function $f(t)$ to transform $s(y_1,y_1)$.

Theorem 7.1 : The Mellin transform of the (Theorem 6.1) gives the following result:

$$\begin{aligned}
&\mathscr{M}\left\{P^{\psi,\xi}_{0,w;r}\left((t^{\eta-1})\,{}^{\varphi}_{p}M^{\zeta}_{q}\left(ax^{\Delta}+1\right)\right);s\right\}=s^{-\eta}\sum_{k=0}^{\infty}\frac{(d_1)_k\ldots(d_p)_k}{(e_1)^k\ldots(e_q)^k}\frac{\Gamma(\eta+\xi k)}{\Gamma(\varphi k+\zeta)}\mathscr{R}(s+\eta-1)\\
&\times I^{\delta,\aleph+2}_{p+2,q+1;r}\left[\rho\left|\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\mu+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\mu+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right.\right]
\end{aligned}\tag{67}$$

$\mathscr{M}(t^{\eta-1};s)$This integral evaluates to $\Gamma(s+\eta-1)$ provided $\mathscr{R}(s+\eta-1)>0$.

7.1. Example Using a Simpler Holomorphic Function:

Consider a simpler function $(f(t)=t^{\alpha}e^{-\beta t})$, where (α) and (β) are constants, and this function is holomorphic (analytic) in the complex plane except possibly at singularities related to (α) and (β).

Let's compute the Mellin transform of this function $(f(t))$:

$$\mathscr{M}[t^{\alpha}e^{-\beta t};s]=\int_0^{\infty}t^{\alpha}e^{-\beta t}t^{s-1}\,dt=\int_0^{\infty}t^{s+\alpha-1}e^{-\beta t}\,dt$$

This integral is a standard gamma function representation:

$$\mathscr{M}[t^{\alpha}e^{-\beta t};s]=\frac{\Gamma(s+\alpha)}{\beta^{s+\alpha}}$$

This result is holomorphic in (s) as long as $(\mathscr{R}(s+\alpha)>0)$, meaning it is analytic (holomorphic) in this region of the complex plane.

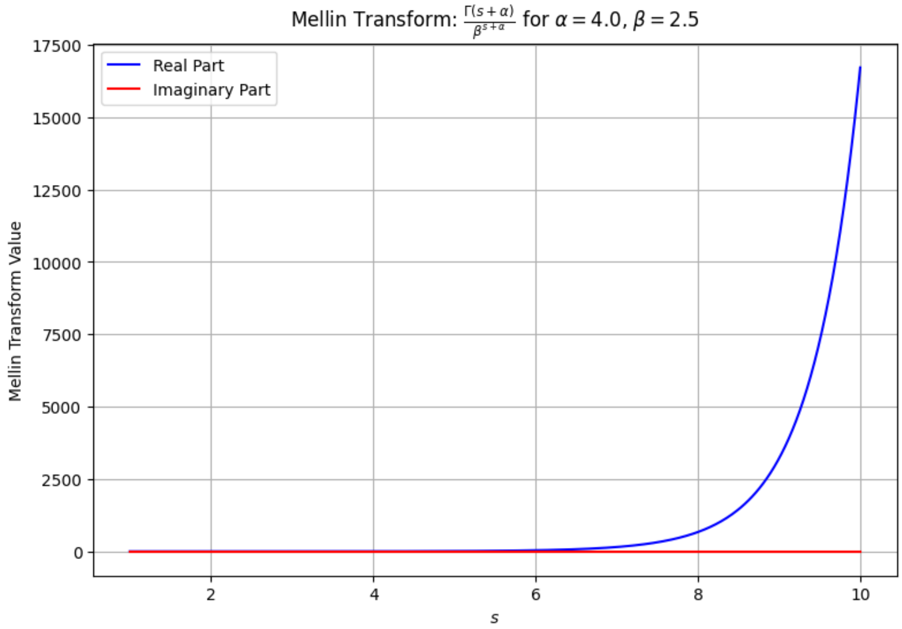

Figure 3: The real component of the Mellin Transform becomes dominant and increases rapidly for large s.

Figure 3, highlights the Mellin Transform's exponential growth, where the real part (blue curve) rapidly increases for larger s, emphasizing its role in asymptotic analysis. The imaginary part (red curve) remains near zero, indicating a predominantly real-valued function. Widely applied in statistical physics, signal processing, and probability theory. The Mellin Transform is essential for asymptotic expansions, fractional calculus, and special function analysis. This visualization provides key insights into the real-dominant behavior of gamma-based transforms and their dependence on s.

Theorem 7.2 : The Mellin transform of the (Theorem 6.2) gives the following result:

$$\mathscr{M}\left\{P^{\phi,\xi}_{x,\infty;r}\left((t^{-\eta})\,{}^{\varphi}_{p}M^{\zeta}_{q}\left(\frac{a}{t^{\Delta}}+1\right)\right);s\right\} = x^{-\eta}\sum_{k=0}^{\infty}\frac{(d_1)_k\ldots(d_p)_k}{(e_1)^k\ldots(e_q)^k}\frac{\mathscr{R}(s-\eta)}{\Gamma(\varphi k+\zeta)}$$
$$\times I^{\delta,\aleph}_{p+2,q+1;r}\left[\rho\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,\aleph} & (a_{ji},\alpha_{ji})_{\aleph+1;p_i} & \left(1-\frac{\phi+\eta+\Delta k}{r},\tau\right) & (-\xi,v^2)\\ \left(-\xi-\frac{\phi+\eta+\Delta k}{r},\tau+v^2\right) & (b_j,\beta_j)_{1,\delta} & (a_{ji},\beta_{ji})_{\delta+1;q_i} & \end{matrix}\right] \tag{68}$$

$\mathscr{M}(t^{-\eta};s)$This integral evaluates to $\Gamma(s-\eta)$ provided $\mathscr{R}(s-\eta)>0, s>0, s>\eta$. [23]

Conditions for Holomorphy:

The gamma function $\Gamma(s-\eta)$ is holomorphic (analytic) in the complex plane except at its poles, which occur at non-positive integers of $s-\eta$. Therefore, the Mellin transform $\Gamma(s-\eta)$ is holomorphic in the region $\mathscr{R}(s)>\eta$. This means the function is holomorphic wherever $\Gamma(s-\eta)$ is defined.

Conditions for Holomorphy:

The Gamma function $\Gamma(z)$ is holomorphic for all z except for non-positive integers. Thus, for $\Gamma(s-\eta)$ to be defined and holomorphic, we require:

$$\mathbb{R}(s-\eta)>0 \implies \mathbb{R}(s)>\eta$$

.

7.2. Example with Specific Parameters:

Lets set $\eta=9$. The Mellin transform of $f(t)=t^{-9}$ becomes:

$$\mathscr{M}[t^{-9};s] = \Gamma(s-9)$$

.

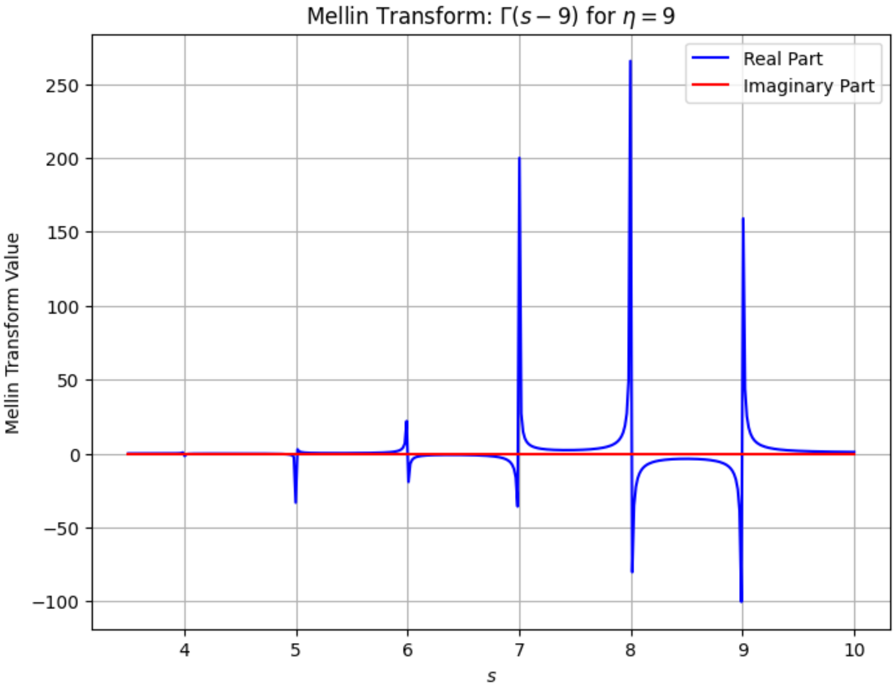

Figure 4: The graph illustrates the pole structure of the gamma function featuring vertical asymptotes at integer values of $s \leq 9$.

Figure 4, illustrates the complex behavior of the Mellin Transform of the Gamma function, particularly its periodic singularities. The sharp spikes occur at integer values of s, which correspond to poles of the Gamma function. The increasing magnitude of the spikes as s increases reflects the rapid growth of the Gamma function for large arguments.

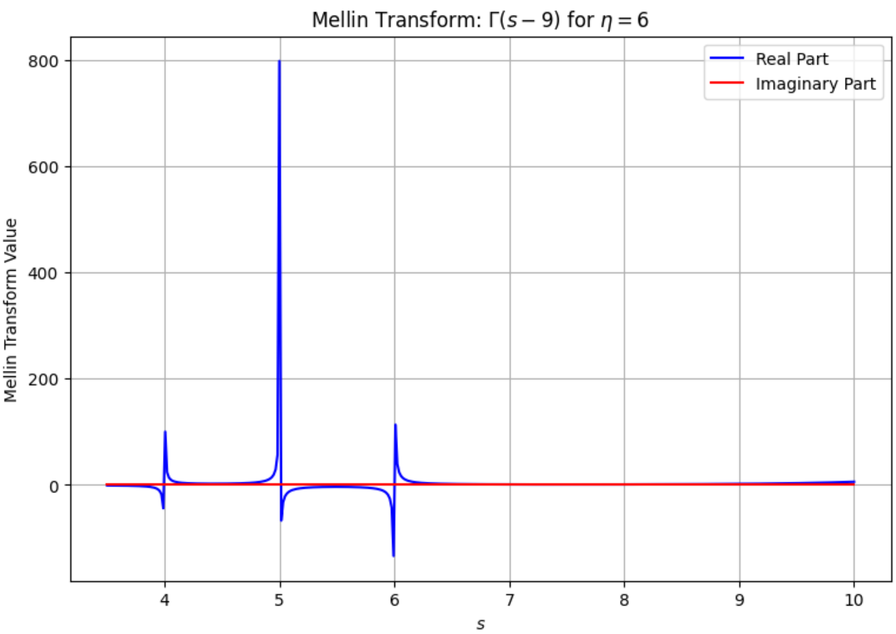

Figure 5: The plot shows the gamma function's pole structure with singularities at specific integer s values.

Figure 5, highlights the pole structure of the gamma function, with vertical spikes marking singularities at integer values where the function diverges. The real part (blue curve) is dominant exhibiting rapid growth near these poles while the imaginary part (red curve) remains small indicating the function is largely real-valued. In complex analysis and physics. The Mellin Transform plays a key role in signal processing, quantum field theory, and asymptotic analysis. The poles are essential for contour integration and residue calculations, making this visualization useful for understanding the gamma functions behavior in the Mellin domain.

8. Conclusions

The study of the generalized M-series in relation to the Meijer G-function and its transformation into the generalized Mittag-Leffler function highlights its significant role in advanced mathematical theory. The integration of fractional integral operators further broadens its applicability, particularly within the realm of fractional calculus. Additionally, the inclusion of the I-function enriches the analytical framework by unifying and extending various special functions. The profound connection between these functions and the Mellin transform offers deeper insights into their structural properties and integral representations reinforcing their theoretical and computational relevance.

The use of Fourier transforms with integral operators presents both challenges and advantages compared to the Mellin transform. One key challenge lies in the fundamental difference in their kernel structureswhile the Mellin transform is naturally suited for power-law behaviors, the Fourier transform emphasizes frequency components, affecting the behavior and convergence of integral operators. Additionally, Fourier transforms may encounter issues with divergence and singularities, necessitating careful regularization. The complexity further increases when combined with fractional integral operators, making analytical solutions more challenging. However, this approach also offers significant advantages. The Fourier transform has broad applicability in signal processing, physics, and engineering, making its integration with integral operators valuable for real-world problems. It also provides direct spectral insights, enabling applications in wave propagation and stability analysis. Moreover, Fourier-based integral operators offer flexibility in solving differential equations, particularly in boundary value problems, where Mellin-based techniques may be less effective.

Funding here: No Funding
Data Availability: All data available in the manuscript.
Conflicts of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References:

[1] Abu-Shady, M., and Kaabar, M. K. (2021). A generalized definition of the fractional derivative with applications. Mathematical Problems in Engineering, 2021(1), 9444803.

[2] Agarwal, P., Cetinkaya, A., Jain, S., and Kiymaz, I. O. (2019). S-Generalized Mittag-Leffler function and its certain properties. Mathematical Sciences and Applications E-Notes, 7(2), 139-148.

[3] Agarwal, P., Rogosin, S. V., Karimov, E. T., and Chand, M. (2015). Generalized fractional integral operators and the multivariable H-function. Journal of Inequalities and Applications, 350(2015), 1-17.

[4] Chaurasia, V. B. L., and Pandey, S. C. (2008). On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions. Astrophysics and space science, 317, 213-219.

[5] Erdlyi, A. (1953). Higher transcendental functions. Higher transcendental functions, 59.

[6] Getz, W. M., and Owen-Smith, N. (2011). Consumer-resource dynamics: quantity, quality, and allocation. PLoS ONE, 6(1), e14539.

[7] Hilfer, R. (Ed.). (2000). Applications of fractional calculus in physics. World Scientific.

[8] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. (2006). Theory and applications of fractional differential equations, Elsevier, 204.

[9] Kilbas, A. A., Saigo, M., and Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms and Special Functions, 15(1), 31-49.

[10] Kilbas, A. A., and Saigo, M. (1996). On Mittag-Leffler type function, fractional calculus operators, and solutions of integral equations. Integral Transforms and Special Functions, 4(4), 355-370.

[11] Kiryakova, V. S. (1993). Generalized fractional calculus and applications. CRC Press.

[12] Kiymaz, I. O., etinkaya, A., and Sahin, R. (2019). Generalized M-series and its certain properties. Journal of Inequalities and Special Functions, 10, 7888.

[13] Mathai, A. M., Saxena, R. K., and Haubold, H. J. (2009). The H-function: theory and applications. Springer Science & Business Media.

[14] Parihar, P. S., and Ali, M. F. (2020). Analysis of some I-function relation. Tathapi (UGC Care Journal), 19(9), 308.

[15] Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Elsevier, 198.

[16] Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Mathematical Journal, 19(1), 7-15.

[17] Saigo, M. (1978). A remark on integral operators involving the Gauss hypergeometric functions, 135-143.

[18] Saigo, M. (1980). A certain boundary value problem for the Euler-Darboux equation. II.

[19] Saigo, M., Saxena, R. K., and Ram, J. (1992). On the fractional calculus operator associated with the H-function. Ganita Sandesh, 6(1), 36-47.

[20] Salim, T. O. (2009). Some properties relating to the generalized Mittag-Leffler function. Adv. Appl. Math. Anal, 4(1), 21-30.

[21] Saxena, R. K., Mathai, A. M., and Haubold, H. J. (2006). Fractional reaction-diffusion equations. Astrophysics and space science, 305, 289-296.

[22] Scalas, E., Gorenflo, R., and Mainardi, F. (2000). Fractional calculus and continuous-time finance. Physica A: Statistical Mechanics and its Applications, 284(1-4), 376-384.

[23] Shah, S. A. H., Mubeen, S., Rahman, G., and Younis, J. (2021). Relation of Some Known Functions in terms of Generalized Meijer G-Functions. Journal of Mathematics, 2021(1), 7032459.

[24] Shukla, A., Shekhawat, S., and Modi, K. (2020). Generalized fractional calculus operators involving multivariable Aleph function. International Journal of Mathematics Trends and Technology-IJMTT, 66.

[25] Srivastava, H. M., Bansal, M. K., and Harjule, P. (2024). A class of fractional integral operators involving a certain general multiindex Mittag-Leffler function. Ukrainian Mathematical Journal, 75(8), 1255-1271.

[26] Zhou, Y. (2023). Basic theory of fractional differential equations. World Scientific.