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Analysis and Numerical Treatment of Chain Smokers using Advanced Fractional Operators

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Abstract:

In this paper, a fractal fractional derivative is used to examine an environmentally friendly approach of understanding the workings of smoking in humans. We proposed a fractional differential equation system to represent a time-fractional order smoking model with illness effects. Studies are conducted using methodologies. Lipschitz circumstances and linear growth are utilized to demonstrate the presence and distinction of the suggested model in relation to the impact of the global offset. It is confirmed that the fractional order model's solutions are bounded and positive. When conducting the initial and subsequent derivative assessments, the Lyapunov function is employed to verify analysis of global stability. In order to examine the influence of smoking on humans, the fractional operator is studied. To do this, solutions are constructed applying the extended version of the Mittag-Leffler kernel using a two-step Lagrange polynomial method. Numerical simulation is performed to see how the fractional order smoking model behaves. Such research will aid in understanding the behavior of smoking and in developing human defenses.

Keywords: Solution Boundedness; Non-Local Kernel; Mathematical Modeling.

1. Introduction

In the eleventh century, Fibonacci applied mathematics to biology for the first time, using his well-known Fibonacci sequence to explain population growth. D Bernoulli explained the impact of smallpox using mathematical methods. Johannes Reinke was the main user of biological mathematics in 1901. Its goal is to model and represent biological processes mathematically. Additionally, it is employed to identify occurrences in living things. Over the past few decades, bio maths has advanced significantly and in the decades to come this advancement will continue. Although math has always been important. In the field of natural science, it will soon be even more so in biology. Bio math fundamentals ought to be taught at a young age. There aren't many basic steps in mathematical biology. After then, anybody can use mathematical models to investigate the biological subject in more detail [1].

Esoteric mathematics and mathematical modeling are included in the field of mathematics. With the aid of mathematical concepts and theory, it is simple to measure the work flow procedure predictions, and results. As a result, biologists rely heavily on mathematics today. Many talented scientists work on mathematical modeling of the biological sciences [2]. The biological system, complex systems that explain structural changes, and differential equations of integer order that depict their dynamics are related to basic mathematical models. In mathematical modeling, the interdependence of parameters is described by the nonlinearity and multi-scale behaviors [3]. Using classical derivatives, numerous biological models have been thoroughly examined in recent decades [4]. The main issue affecting healthy communities worldwide is smoking. Smoking affects several body organs and is the cause of over a million deaths worldwide. For smokers, The chance of having a heart attack is 70%. More as in contrast to non-smokers. Similarly, the incidence of lung cancer is 10% higher in smokers compared to non-smokers. Short-term smokers' primary bad breath is one of the side effects, high BP discolored teeth and cough. Oral illness and stomach ulcers, heart disease,Mouth cancer and throat cancer are also prevalent among smokers and lung cancer are the main side effects of long-term smoking that have been reported in recent years. Additionally, smokers live between 12% and 13% years less than non-smokers. Studied the Model of smoking connected to the fractional derivative of Caputo [5]. Smoking will kill 10 million people worldwide, according to the WHO. The death rate is greater than that of all other diseases. The tobacco user fourteen years earlier in life than a person who does not smoke [6]. The primary cause of cancer and other diseases is tobacco use. In poor nations, diseases linked to tobacco use claimed the lives of around 70% of people[7]. Smoking is a major health concern that people today face. According to the (WHO), an assessment of the smoking pandemic, a large number of smokers die in their prime working years [8].

Approximately $440,000$ deaths in the US and $105,000$ deaths in the UK are attributed to smoking-related illnesses each year. Over 8 millions of people pass away. From the global tobacco pandemic each year, with around 1.2 million fatalities from second hand smoke exposure, putting it among the greatest threats to public health the world has ever faced [9]. Reducing smoking rates among individuals is a goal shared by all medical professionals and researchers. To better understand the dynamics of smoking and contribute to the decrease in the number of smokers, There are developed mathematical models in conjunction with experimental investigation. To recognize and mitigate the harm that smoking causes to public well ness, [10] performed a processing analysis of a Partially ordered smoking model utilizing an inexpensive, highly accurate cyclical method. Ucar et al. [11] investigated the smoking model's behavior and its implications for well-being through an extensive analysis that made use of the Atangana-Baleanu derivative. Khan et al. [12] the application of harmonic type incidence rate in the provided smoking model, emphasizing the thorough application of fractal and fractional calculus to represent real-life dynamics. Melkamu and Mebrate [13] suggested a fractional smoking model to study smoking-related difficulties in the actual world by taking into account second-hand and third-hand smokers. Because these new definitions feature nonsingular kernels that are tailored to their demands, they have had a significant influence.

Caputo fractional derivatives [14] and AB [15], the sole reason why the fractional derivatives have varied from one another is that ABis determined by a power law, Fabrizio by an exponential decay law, and Caputo by an ML law. The function of differential equations of any order is deliberated by Bulut et al. [16]. Kilbas et al, provide an explanation of the fundamental ideas of fractional differential equations and how to use them [17]. The model of Keller-Segel concerning a fractional derivative with a kernel that is not singular was studied by Atangana and Koca [18]. Huang et al, have introduced fractional logistic maps [19]. Zaman conducted research on the qualitative reaction of the dynamics of quitting smoking [20]. Singh et al, investigation uses the quitting smoking model connected with the Caputo fractional derivative[21]. Examined the smoking model's connection to Caputo's fractional derivative [22]. Examined the best way to control the smoking models and provided a qualitative study of the smoking dynamics [23]. An analysis of lung cancer and cigarette smoking [24]. Explain the mathematical study of tobacco's dynamics, including its recovery and decrease [25]. Described the smoking-related fractional mathematical model [26]. Outlined the dynamics of smoking cessation [27]. Investigated a fractional smoking among numerous others[28]. Interpret the mathematical system of equations description of smoking's global dynamics [29]. Researchers have determined the rate at which smoking is expanding by classifying smoker. Castillo-Garrsow et al presented in 1997, a broad epidemic survey [30] to explain the dynamic features of the smoking cessation paradigm. Three groups of people were assumed by the researchers to exist prospective smokers (P), smokers as of right now (S), and former smokers (Q). They are also considered the potential effects of therapy, counseling, relapse and peer pressure on individuals [31]. It gave rise to a brand-new group of smokers known as Qt-smokers who briefly gave off smoking in 2008.

Presenting a dynamic interaction with an integer order and a new class of smokers that smoke occasionally [32] enlarged the model [33] created a nonlinear a mathematical model for examination the movement of smoking spread in the world. The model used Lyapunov functions are used to evaluate overall steadiness, mutual occurrence of equilibrium and stability overall, [34]. Demonstrated a smoking model that looked into the stability, best control strategy and qualitative behavior of diseases smoking's influence. Using the fractional Atangana-Baleanu derived [35] suggested an epidemiological methodology to simulate the dynamics of the insurgent population and emphasized the need of reducing radicalization and focusing on recovery rates in order to have a successful counterinsurgency tactics. Using equilibrium stability analysis, [36] investigated how health education programs affect the dynamics of smoking and showed how they can lower the number of smokers. A computational representation of The topic of secondhand smoking is examined by [37]. In addition to a numerical simulation analysis of its sensitivity and stability [38]. Examined an intervention-based model of non-integer smoking techniques that showed how effective the best control measures were at lowering vulnerable individuals and smokers. Other similar models include the epidemiological smoking model [39]. Model of vector-borne illness [40], model of invasion and spread of the tumor [41]. The COVID-19 model of propagation delay [42], model of COVID-19 transmission [43], dynamics of TB transmission model [44]. Relative differential is ability to control inclusions in delays [45], Hilfer fractional differential system controllability [46] were offered in both fractional and integer order by numerous authors.

2. Formulation of the model

This section presents the biological model of smoking. The entire people $N(\eta)$ is divided into 5 classifications: smokers $S(\eta)$, Infrequent smokers $O(\eta)$, potential smokers $P(\eta)$, and Smokers with a temporary habit. Both permanent $Q_p(\eta)$ quitting and $Q_t(\eta)$ quitting. Hence, where η is time, can be expressed as. The following are governing equations that are nonlinear for the smoking model (as explained by Takasar et al., 43]): where μ is the Rate at which potential smokers are recruited, α is the rate of successful interactions between P and S, ν is the rate of natural mortality, β_1 is the Conversion rate of O to S, β_2 is the transformation rate of Q_t to S, δ is the rate of smoker cessation, ε is the disease-related death rate, and 1-γ is the rate of temporary quitting.

$$\begin{aligned}
P'(\eta) &= \mu - \alpha P(\eta)S(\eta) - \nu P(\eta) && (1)\\
O'(\eta) &= \alpha P(\eta)S(\eta) - \beta_1 O(\eta) - \nu O(\eta) && (2)\\
S'(\eta) &= \beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\nu+\delta+\varepsilon) && (3)\\
Q_t'(\eta) &= -\beta_2 S(\eta)Q_t(\eta) + \delta(1-\gamma)S(\eta) - \nu Q_t(\eta) && (4)\\
Q_p'(\eta) &= \gamma\delta S(\eta) - \nu Q_p(\eta). && (5)
\end{aligned}$$

Now we changes above equation into FFM respectively,

$$\begin{aligned}
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= \mu - \alpha P(\eta)S(\eta) - \nu P(\eta) && (6)\\
0^{FFM}D_t^{\zeta,\tau}O'(\eta) &= \alpha P(\eta)S(\eta) - \beta_1 O(\eta) - \nu O(\eta) && (7)\\
0^{FFM}D_t^{\zeta,\tau}S'(\eta) &= \beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\nu+\delta+\varepsilon) && (8)\\
0^{FFM}D_t^{\zeta,\tau}Q_t'(\eta) &= -\beta_2 S(\eta)Q_t(\eta) + \delta(1-\gamma)S(\eta) - \nu Q_t(\eta) && (9)\\
0^{FFM}D_t^{\zeta,\tau}Q_p'(\eta) &= \gamma\delta S(\eta) - \nu Q_p(\eta), && (10)
\end{aligned}$$

where $P(0)=\psi_1$, $O(0)=\psi_2$, $S(0)=\psi_3$, $Q_t(0)=\psi_4$ and $Q_p(0)=\psi_5$ and ψ_k where $k=1,2,3,4,5$ are initial state of the system of governance.

Theorem 1: Let the initial conditions be $\{P(0), O(0), S(0), Q_t(0), Q_P(0)\}$. Then, if the solutions $\{P, O, S, Q_t, Q_P\}$ exist, they are all positive for all $t \geq 0$.

Proof: To demonstrate why the answers are superior, let's begin with a primitive analysis. By using the provided method, the answers illustrate real-world problems with positive values. In this section, we examine the conditions necessary to guarantee positive solutions from the proposed model. We will specify the standard criteria required to achieve this.

$$\|a\|_\infty = \sup_{t \in D_a} |a(t)| \tag{11}$$

where D_a is the domain of a. Let start with the $P'(\eta)$

$$\begin{aligned}
{}_o^{FFM}D_t^{\zeta,\tau} P'(\eta) &= \mu - \alpha S(\eta)P(\eta) - \nu P(\eta) \\
&\geq -(\alpha P\|S\| + VP) \\
&\geq -(V + \alpha \sup_{\eta \in D_S} \|S\|)P \\
&\geq -(V + \alpha\|S\|_\infty)P
\end{aligned}$$

this yields

$$P'(\eta) \geq O(0)E_\zeta[-\frac{c^{1-\tau}\zeta(V+\alpha\|S\|_\infty)t^\zeta}{AB(\zeta)-(1-\zeta)(V+\alpha\|S\|_\infty)}] \tag{12}$$

c represents the time component. As a result, $P'(\eta)$ is positive for every $t = 0$

$$\begin{aligned}
0^{FFM}D_t^{\zeta,\tau} O'(\eta) &= \alpha S(\eta)P(\eta) - O(\eta)\beta_1 - O(\eta)\nu \\
&\geq -O(\eta)(\nu + \beta_1)
\end{aligned}$$

this yields

$$O'(\eta) \geq O(0)E_\zeta[-\frac{c^{1-\tau}\zeta(\nu+\beta_1)t^\zeta}{AB(\zeta)-(1-\zeta)(\nu+\beta_1)}] \tag{13}$$

where the time component c is. This demonstrates that $O'(\eta)$ for all $t = 0$

$$\begin{aligned}
0^{FFM}D_t^{\zeta,\tau} S'(\eta) &= \beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\delta+\nu+\varepsilon) \\
&\geq -((\delta+\nu+\varepsilon))S(\eta) \\
&\geq -((\delta+\nu+\varepsilon))S(\eta)
\end{aligned}$$

this yields

$$S'(\eta) \geq O(0)E_\zeta[-\frac{c^{1-\tau}\zeta(\nu+\delta+\varepsilon)t^\zeta}{AB(\zeta)-(1-\zeta)(\nu+\delta+\varepsilon)}] \tag{14}$$

c represents the time component. Hence, $S'(\eta)$ is positive for all $t = 0$

$$\begin{aligned}
0^{FFM}D_t^{\zeta,\tau} Q_t'(\eta) &= -\beta_2 S(\eta)Q_t(\eta) + \delta(1-\gamma)S(\eta) - Q_t(\eta)\nu \\
&\geq -Q_t(\nu + (\beta_2 S(\eta)) \\
&\geq -(\nu + \beta_2\|S\|)Q_t'(\eta) \\
&\geq -(\nu + \beta_2 \sup_{\eta \in D_S} \|S\|)Q_t'(\eta) \\
&\geq -((\nu + \beta_2\|S\|_\infty)Q_t'(\eta)
\end{aligned}$$

this yields

$$Q_t'(\eta) \geq O(0)E_\zeta[-\frac{c^{1-\tau}\zeta(\nu+\beta_2\|S\|_\infty)t^\zeta}{AB(\zeta)-(1-\zeta)(\nu+\beta_2\|S\|_\infty)}] \tag{15}$$

c represents the time component. According to this, $Q_t'(\eta)$ is positive for all t = 0

$$\begin{aligned} 0^{FFM}D_t^{\zeta,\tau}Q_p'(\eta) &= \gamma\delta S(\eta)-\nu Q_p(\eta) \\ &\geq -(\nu)Q_p(\eta) \end{aligned}$$

this yields

$$Q_p'(\eta) \geq O(0)E_\zeta[-\frac{c^{1-\tau}\zeta(\nu)t^\zeta}{AB(\zeta)-(1-\zeta)(\nu)}] \tag{16}$$

here c is a component of time. This shows that $Q_p'(\eta)$ is advantageous to everyone t = 0.

Theorem 2: Models $6-10$ have bounded solutions for all positive initial conditions.

Proof: Theorem 1 demonstrates that the model's solutions are positive $\forall\, t \geq 0$, and methods given As $X = P+O+S$. So, we have

$$\begin{aligned} 0^{FFM}D_t^{\zeta,\tau}X(\eta) &= \{\mu-\upsilon(X)-S(\delta+\varepsilon-\beta_2 Q_t)\} \\ weobtain & \\ E_p &= \{P,S,O\in R_+^3|S+O\leq X\} \end{aligned} \tag{17}$$

Further we have $X_\upsilon = Q_t+Q_p$ so we have

$$0^{FFM}D_t^{\zeta,\tau}X_\upsilon = -\beta_2 SQ_t+\delta S-\upsilon(X_\upsilon)$$

After solving we get

$$E_\upsilon \leq \frac{\delta S-\beta_2 SQ_t}{\upsilon}$$

Thus

$$E_\upsilon = \{Q_t,Q_p\in R_+^2|X_\upsilon \leq \frac{\delta S-\beta_2 SQ_t}{\upsilon}\}$$

The solutions of the model (3.1.6)(3.1.110) are confined to region E.

$$E = \{P,S,O,Q_t,Q_p\in R_+^5|S+O\leq X, X_\upsilon \leq \frac{\delta S-\beta_2 SQ_t}{\upsilon}\}$$

which demonstrates that under the specified initial conditions in the region, all of the solutions remain positive invariant. E for all $t=0$

Theorem 3: Apart from the initial case, the proposed plant virus model is unique and bounded in R_+^5.

Proof: Here, we used the described process. We own

$$\begin{aligned} 0^{FFM}D_t^{\zeta,\tau}P'((\eta))_{P=0} &= \mu,\geq 0 \\ 0^{FFM}D_t^{\zeta,\tau}S'((\eta))_{S=0} &= \alpha PS,\geq 0 \\ 0^{FFM}D_t^{\zeta,\tau}O'((\eta))_{O=0} &= \beta_1 O,\geq 0 \\ 0^{FFM}D_t^{\zeta,\tau}Q_t'((\eta))_{Q_t=0} &= \delta(1-\gamma)S,\geq 0 \\ 0^{FFM}D_t^{\zeta,\tau}Q_p'((\eta))_{Q_p=0} &= \gamma\delta(S),\geq 0 \end{aligned} \tag{18}$$

If $(P(0),S(0),O(0),Q_t(0),Q_p(0)) \in \mathbb{R}_+^5$, then the solution cannot leave the hyperplane. This demonstrates the positive invariant set nature of the domain $\mathbb{R}_+^5$.

3. Effect of globel derivative:

It has long been established in the literature that the RiemannStieltjes integral, of which the classical integral is a special case, is one of the most frequently encountered integrals. If

$$Z(x) = \int z dx \tag{19}$$

The Riemann Stieltjes integral is

$$Z_w(x) = \int z(x) dx \tag{20}$$

The global derivative of $z(x)$ with respect to $w(x)$ is

$$D_w z(x) = \lim_{h \longrightarrow 0} \frac{z(x+h) - z(x)}{w(x+h) - w(x)} \tag{21}$$

if both function can be differentiated classically, then

$$D_w z(x) = \frac{z^{'}(x)}{w^{'}(x)}, \tag{22}$$

providing that $w'(x) \neq 0, \forall x \varepsilon D_{g'}$ '. We will apply this concept in this section to see if it affects the plant virus model. We will accomplish this by replacing the classical derivative with worldwide derivative.

$$\begin{aligned}
D_w P &= \mu - \alpha P(\eta) S(\eta) - \nu P(\eta) \\
D_w O &= \alpha P(\eta) S(\eta) - \beta_1 O(\eta) - \nu O(\eta) \\
D_w S &= \beta_1 O(\eta) + \beta_2 S(\eta) Q_t(\eta) - (\nu + \delta + \varepsilon) \\
D_w Q_t &= -\beta_2 S(\eta) Q_t(\eta) + \delta(1-\gamma) S(\eta) - \nu Q_t(\eta) \\
D_w Q_p &= \delta \gamma S(\eta) - \nu Q_p(\eta)
\end{aligned} \tag{23}$$

It will be assumed for simplicity's sake that g is differentiable, so ,

$$\begin{aligned}
P' &= w'[\mu - \alpha P(\eta) S(\eta) - \nu P(\eta)] \\
O' &= w'[\alpha P(\eta) S(\eta) - \beta_1 O(\eta) - \nu 0(\eta)] \\
S' &= w'[\beta_1 O(\eta) + \beta_2 S(\eta) Q_t(\eta) - (\nu + \delta + \varepsilon)] \\
Q_t' &= w'[-\beta_2 S(\eta) Q_t(\eta) + \delta(1-\gamma) S(\eta) - \nu Q_t(\eta)] \\
Q_p' &= w'[\gamma \delta S(\eta) - \nu Q_p(\eta)]
\end{aligned} \tag{24}$$

There will be a specific procedure where the function $w(\eta)$ is appropriately selected. If $w(\eta) = \eta^a, a \in R$, for instance, then fractal behavior will be observed.As long as we keep in mind that

$$\|w'\|\_\infty = \sup_{\eta \in D'_w} |w'(\eta)| < N \tag{25}$$

The ensuing example will demonstrate that the equation system is capable of accepting a single solution.

$$\begin{aligned}
P' &= w'[\mu - \alpha P(\eta)S(\eta) - \nu P(\eta)] = Z_1(\eta, P, O, S, Q_t, Q_p) \\
O' &= w'[\alpha P(\eta)S(\eta) - \beta_1 O(\eta) - \nu 0(\eta)] = Z_2(\eta, P, O, S, Q_t, Q_p) \\
S' &= w'[\beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\nu + \delta + \varepsilon)] = Z_3(\eta, P, O, S, Q_t, Q_p) \\
Q'_t &= w'[-\beta_2 S(\eta)Q_t(\eta) + \delta(1-\gamma)S(\eta) - \nu Q_t(\eta)] = Z_4(\eta, P, O, S, Q_t, Q_p) \\
Q'_p &= w'[\gamma\delta S(\eta) - \nu Q_p(\eta)] = Z_5(\eta, P, O, S, Q_t, Q_p)
\end{aligned} \tag{26}$$

In order to do this, we must confirm the next two requirements:

$$1. |Z(\eta, P, O, S, Q_t, Q_p)|^2 < k(1 + |P|^2), \qquad \forall P_1, P_2.$$

We have

$$2. \|Z(\eta, P_1, O, S, Q_t, Q_p) - Z(\eta, P_2, O, S, Q_t, Q_p)\|^2 < k\|P_1 - P_2\|_\infty^2 \tag{27}$$

Initially,

$$|Z_1(\eta, P, O, S, Q_t, Q_p)|^2 = |w'[\mu - \alpha P(\eta)S(\eta) - \nu P(\eta)]|^2$$

$$= \quad |w'[(\mu) + (-\alpha S - \nu)P]|^2$$

$$\leq 2|w'|^2(|\mu|^2 + |-\alpha S - \nu|^2|P|^2)$$

$$\leq 2 \sup_{\eta \in D_{w'}} |w'|^2(\mu)^2 + 4|w'|^2((\alpha^2 \sup_{\eta \in D_s} |S|^2 + \nu^2)P^2)$$

$$\leq 2\|w'\|_\infty^2(\mu^2) \times (1 + \frac{2((\alpha^2\|S\|_\infty^2 + \nu^2)P^2)}{\mu^2})$$

$$\leq k_1(1 + |P|^2)$$

under the condition

$$\frac{2(\alpha^2\|S\|_\infty^2 - \nu^2)}{\mu^2} < 1$$

where

$$k_1 = 2\|w'\|_\infty^2(\mu^2)$$

$$|Z_2(\eta, P, O, S, Q_t, Q_p)|^2 = |w'[\alpha P(\eta)S(\eta) - \beta_1 O(\eta) - \nu 0(\eta)]|^2$$

$$= \quad |w'[(\alpha PS) + (-\beta_1 - \nu)O]|^2$$

$$\leq 2|w'|^2(|\alpha PS|^2 + |(-\beta_1 - \nu)O|^2|O|^2)$$

$$\leq 2\|w'\|_\infty^2(\alpha^2\|P\|_\infty^2\|S\|_\infty^2) \times (1 + \frac{2(\beta_1^2 + \nu^2)O^2}{(\alpha^2\|P\|_\infty^2\|S\|_\infty^2)})$$

$$\leq k_2(1 + |O|^2)$$

under the condition

$$\frac{2(\beta_1^2 + \nu^2)O^2}{(\alpha^2\|P\|_\infty^2\|S\|_\infty^2)}) < 1$$

where

$$k_2 = 2\|w'\|_\infty^2(\alpha^2\|P\|_\infty^2\|S\|_\infty^2)$$

$$Z_3|(\eta, P, O, S, Q_t, Q_p)|^2 = |w'[\beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\nu + \delta + \varepsilon)]|^2$$

$$= \quad |w'[(\beta_1 O + \beta_2 S Q_t) + |-(\nu + \delta + \varepsilon)|^2|S|^2)]$$

$$\leq 4 \sup_{\eta \in D_{w'}} |w'|^2(\beta_1^2 \sup_{\eta \in D_O} |O|^2 + \beta_2^2 \sup_{\eta \in D_S} |S|^2 \sup_{\eta \in D_{Q_t}} |Q_t|^2) + 2 \sup_{\eta \in D_{w'}} |w'|^2(\nu + \delta + \varepsilon) \sup_{\eta \in D_S} |S|^2$$

$$\leq 4\|w'\|_\infty^2(\beta^1\|O\|_\infty^2 + \beta_2\|S\|_\infty^2\|Q_t\|_\infty^2) \times (1 + \frac{1(\nu + \delta + \varepsilon)^2|S|^2}{2(\beta^1\|O\|_\infty^2 + \beta_2\|S\|_\infty^2\|Q_t\|_\infty^2)}$$

$$\leq k_3(1 + |S|^2)$$

under the condition

$$\frac{1(\nu + \delta + \varepsilon)^2}{2(\beta^1\|O\|_\infty^2 + \beta_2\|S\|_\infty^2\|S\|_\infty^2)} < 1$$

where

$$k_3 = 4\|w'\|_\infty^2 2(\beta^1\|O\|_\infty^2 + \beta_2\|S\|_\infty^2\|Q_t\|_\infty^2)$$

$$\begin{aligned}
Z_4|(\eta,P,O,S,Q_t,Q_p)|^2 &= |w'[-\beta_2 S(\eta)Q_t(\eta)+\delta(1-\gamma)S(\eta)-\nu Q_t(\eta]|^2 \\
&= |w'[(\delta S)+(-\gamma\delta S-\beta_2 S-\nu)Q_t]|^2 \\
&\leq |w'|^2[|(\delta S)|^2+|(-\gamma\delta S)(-\beta_2 S-\nu)|^2|Q_t|^2] \\
&\leq 2\sup_{\eta\in D_{w'}}|w'|^2(\delta^2\sup_{\eta\in D_S}|S|^2)+ \\
&\quad (6\sup_{\eta\in D_{w'}}|w'|^2(\gamma^2\delta^2\sup_{\eta\in D_S}|S|^2)+(\beta_2^2\sup_{\eta\in D_S}|S|^2+\nu^2)|Q_t|^2) \\
&\leq 2\|w'\|_\infty^2\delta^2|S|_\infty^2\times(1+\frac{3(\gamma^2\delta^2|S|_\infty^2+\beta_2^2|S|_\infty^2+\nu^2)|Q_t|^2}{\delta^2\sup_{\eta\in D_S}|S|^2}) \\
&\leq k_4(1+|Q_t|^2)
\end{aligned}$$

$$Z_4(\eta,P,O,S,Q_t,Q_p)|^2=|w'[-\beta_2 S(\eta)Q_t(\eta)+\delta(1-\gamma)S(\eta)-\nu Q_t(\eta]|^2$$

$$= \quad |w'[(\delta S)+(-\gamma\delta S-\beta_2 S-\nu)Q_t]|^2$$

$$\leq |w'|^2[|(\delta S)|^2+|(-\gamma\delta S)(-\beta_2 S-\nu)|^2|Q_t|^2]$$

$$\leq 2\sup_{\eta\in D_{w'}}|w'|^2(\delta^2\sup_{\eta\in D_S}|S|^2)+(6\sup_{\eta\in D_{w'}}|w'|^2(\gamma^2\delta^2\sup_{\eta\in D_S}|S|^2)+(\beta_2^2\sup_{\eta\in D_S}|S|^2+\nu^2)|Q_t|^2)$$

$$\leq 2\|w'\|_\infty^2\delta^2|S|_\infty^2\times(1+\frac{3(\gamma^2\delta^2|S|_\infty^2+\beta_2^2|S|_\infty^2+\nu^2)|Q_t|^2}{\delta^2\sup_{\eta\in D_S}|S|^2})$$

$$\leq k_4(1+|Q_t|^2)$$

under the condition

$$\frac{3(\gamma^2\delta^2|S|_\infty^2+\beta_2^2|S|_\infty^2+\nu^2)|Q_t|^2}{\delta^2\sup_{\eta\in D_S}|S|^2}<1$$

where

$$k_4=2\|w'\|_\infty^2\delta^2|S|_\infty^2$$

$$Z_5(\eta,P,O,S,Q_t,Q_p)|^2=|w'[\gamma\delta S(\eta)-\nu Q_p(\eta)]|^2$$

$$\leq 2|w'|^2(|(\gamma\delta S)|^2+(|-\nu Q_p|^2))$$

$$\leq 2\sup_{\eta\in D_{w'}}|w'|^2(\gamma^2\delta^2\sup_{\eta\in D_S}|S|^2)+2\sup_{\eta\in D_{w'}}|w'|^2(\nu^2 Q_p^2)$$

$$\leq 2|w'|_\infty^2(\gamma^2\delta^2|S|_\infty^2)\times(1+\frac{(\nu^2 Q_p^2)}{\gamma^2\delta^2|S|_\infty^2})$$

$$\leq k_5(1+|Q_p|^2)$$

under the condition

$$\frac{(\nu^2 Q_p^2)}{\gamma^2\delta^2|S|_\infty^2} < 1$$

where

$$k_5 = 2|w'|_\infty^2(\gamma^2\delta^2|S|_\infty^2)$$

Therefore, the condition for linear growth is met. Additionally, we confirm the Lipschitz condition. if

$$|Z_1(\eta,P_1,O,S,Q_t,Q_p) - |Z_1(\eta,P_2,O,S,Q_t,Q_p)|^2 = |w'(-\alpha S-\nu)(P_1-P_2)|^2$$

$$|Z_1(\eta,P_1,O,S,Q_t,Q_p) - |Z_1(\eta,P_2,O,S,Q_t,Q_p)|^2 \leq |w'|^2(2\alpha^2|S|^2+2\nu^2)|(P_1-P_2)|^2$$

$$|Z_1(\eta,P_1,O,S,Q_t,Q_p) - |Z_1(\eta,P_2,O,S,Q_t,Q_p)|^2 \leq \sup_{\eta\in D_{w'}}|w'|^2(2\alpha^2\sup_{\eta\in D_S}|S|^2)\times\sup_{\eta\in D_P}|P_1-P_2|^2$$

$$|Z_1(\eta,P_1,O,S,Q_t,Q_p) - |Z_1(\eta,P_2,O,S,Q_t,Q_p)|_\infty^2 \leq \|w'\|_\infty^2(2\alpha^2\|S\|_\infty^2)|P_1-P_2|_\infty^2$$

$$|Z_1(\eta,P_1,O,S,Q_t,Q_p) - |Z_1(\eta,P_2,O,S,Q_t,Q_p)|_\infty^2 \leq \bar{k}_1|P_1-P_2|_\infty^2$$

where

$$\bar{k}_1 = \|w'\|_\infty^2(2\alpha^2\|S\|_\infty^2)$$

$$|Z_2(\eta,P,O_1,S,Q_t,Q_p) - |Z_2(\eta,P,O_2,S,Q_t,Q_p)|^2 = |w'(-(\beta_1+\nu))(O_1-O_2)|^2$$

$$|Z_2(\eta,P,O_1,S,Q_t,Q_p) - |Z_2(\eta,P,O_2,S,Q_t,Q_p)|^2 \leq |w'|^2(2\beta_1^2+2\nu^2)|(O_1-O_2)|^2 \tag{28}$$

$$|Z_2(\eta,P,O_1,S,Q_t,Q_p) - |Z_2(\eta,P,O_2,S,Q_t,Q_p)|^2 \leq \sup_{\eta\in D_{w'}}|w'|^2(2\beta_1^2+2\nu^2)\times\sup_{\eta\in D_P}|O_1-O_2|^2$$

$$|Z_2(\eta,P,O_1,S,Q_t,Q_p) - |Z_2(\eta,P,O_2,S,Q_t,Q_p)|_\infty^2 \leq \|w'\|_\infty^2(2\beta_1^2+2\nu^2)|O_1-O_2|_\infty^2$$

$$|Z_2(\eta,P,O_1,S,Q_t,Q_p) - |Z_2(\eta,P,O_2,S,Q_t,Q_p)|_\infty^2 \leq \bar{k}_2|O_1-O_2|_\infty^2$$

where

$$\bar{k}_2 = \|w'\|_\infty^2(2\beta_1^2+2\nu^2)$$

$$|Z_3(\eta,P,O,S_1,Q_t,Q_p) - |Z_3(\eta,P,O,S_2,Q_t,Q_p)|^2 = |w'(\beta_2 Q_t-(\nu+\delta+\varepsilon)(S_1-S_2))|^2$$

$$|Z_3(\eta,P,O,S_1,Q_t,Q_p)-|Z_3(\eta,P,O,S_2,Q_t,Q_p)|^2 \leq \tag{29}$$
$$|w'|^2(2\beta_2^2|Q_t|^2+6(\nu^2+\delta^2+\varepsilon^2)) \tag{30}$$
$$|(S_1-S_2)|^2$$

$$|Z_3(\eta,P,O,S_1,Q_t,Q_p)-|Z_3(\eta,P,O,S_2,Q_t,Q_p)|^2 \leq \tag{31}$$
$$\sup_{\eta\in D_{w'}}|w'|^2(2\beta_2^2\sup_{\eta\in D_c} \tag{32}$$
$$|Q_t|^2+6(\nu^2+\delta^2+\varepsilon^2))$$
$$\times\sup_{\eta\in D_S}|S_1-S_2|^2$$

$$|Z_3(\eta,P,O,S_1,Q_t,Q_p)-|Z_3(\eta,P,O,S_2,Q_t,Q_p)|^2_\infty \leq \tag{33}$$
$$\|w'\|2(\beta_2^2\|Q_t\|^2_\infty+6(\nu^2+\delta^2+\varepsilon^2))$$
$$\times\sup_{\eta\in D_S}|S_1-S_2|^2$$

$$|Z_3(\eta,P,O,S_1,Q_t,Q_p)-|Z_3(\eta,P,O,S_2,Q_t,Q_p)|^2_\infty \leq \bar{k}_3|S_1-S_2|^2_\infty$$

where

$$\bar{k}_3=\|w'\|2(\beta_2^2\|Q_t\|^2_\infty+6(\nu^2+\delta^2+\varepsilon^2))$$

$$|Z_4(\eta,P,O,S,Q_{1t},Q_p)-Z_4(\eta,P,O,S,Q_{2t},Q_p)|^2=|w'(-\beta_2 S-\nu)(Q_{1t}-Q_{2t})|^2$$

$$|Z_4(\eta,P,O,S,Q_{1t},Q_p)-Z_4(\eta,P,O,S,Q_{2t},Q_p)|^2\leq|w'|^2(\beta_2^2|S|^2-\nu^2)|(Q_{1t}-Q_{2t})|^2$$

$$|Z_4(\eta,P,O,S,Q_{1t},Q_p)-Z_4(\eta,P,O,S,Q_{2t},Q_p)|^2$$
$$\leq\sup_{\eta\in D_{w'}}|w'|^2(\beta_2^2\sup_{\eta\in D_S}|S|^2-\nu^2)$$
$$\times\sup_{\eta\in D_{Q_t}}|(Q_{1t}-Q_{2t})|^2$$

$$|Z_4(\eta,P,O,S,Q_{1t},Q_p)-Z_4(\eta,P,O,S,Q_{2t},Q_p)|^2_\infty\leq|w'|^2_\infty(\beta_2^2|S|^2_\infty-\nu^2)$$
$$\times|(Q_{1t}-Q_{2t})|^2_\infty$$

$$|Z_4(\eta,P,O,S,Q_{1t},Q_p)-|Z_4(\eta,P,O,S,Q_{2t},Q_p)|^2_\infty\leq\bar{k}_4|(Q_{1t}-Q_{2t})|^2_\infty$$

where

$$\bar{k}_4=|w'|^2_\infty(\beta_2^2|S|^2_\infty-\nu^2)$$

$$|Z_5(\eta,P,O,S,Q_t,Q_{1p})-Z_5(\eta,P,O,S,Q_t,Q_{2p})|^2=|w'(-\nu))(Q_{1p}-Q_{2p})|^2$$

$$|Z_5(\eta,P,O,S,Q_t,Q_{1p})-Z_5(\eta,P,O,S,Q_t,Q_{2p})|^2 \leq (\nu^2))|(Q_{1p}-Q_{2p})|^2$$

$$|Z_5(\eta,P,O,S,Q_t,Q_{1p})-Z_5(\eta,P,O,S,Q_t,Q_{2p})|^2 \leq \sup_{\eta\in D_{w'}}|w'|^2(\nu^2) \times \sup_{\eta\in D_{Q_p}}|(Q_{1p}-Q_{2p})|^2$$

$$|Z_5(\eta,P,O,S,Q_t,Q_{1p})-Z_5(\eta,P,O,S,Q_t,Q_{2p})|_\infty^2 \leq |w'|_\infty^2(\nu^2) \times|(Q_{1p}-Q_{2p})|_\infty^2$$

$$|Z_5(\eta,P,O,S,Q_t,Q_{1p})-Z_5(\eta,P,O,S,Q_t,Q_{2p})|_\infty^2 \leq \bar{k_5}|(Q_{1p}-Q_{2p})|_\infty^2$$

where

$$\bar{k_5}=|w'|_\infty^2(\nu^2)$$

4. Analysis of equilibrium points:

This section presents an analysis of equilibrium points.In order to determine the equilibrium points, we must set the system's left side, $1-5$, to 0. For this model, the disease-free equilibrium is

$$K_1(P,O,S,Q_t,Q_p)=(\frac{\mu}{\nu},0,0,0,\frac{\gamma\delta}{\nu}) \tag{34}$$

5. Global stability Analysis:

The global stability analysis employs Lyapunov's technique and Lasalle's invariance concept to determine disease eradication conditions.

5.1. First derivative of Lyapunov

Theorem:3.4. The endemic equilibrium points of the (P,O,S,Q_t,Q_p) model are globally asymptotically stable when the reproductive number $R_0>1$.

Proof: To prove this, a Lyapunov function can be defined as follows:

$$\begin{aligned} L(P^*O^*S^*Q_t^*Q_p^*) = \quad & (P-P^*-P^*\log\tfrac{P}{P^*}) \\ & +(O-O^*-O^*\log\frac{O}{O^*}) \\ & +(S-S^*-S^*\log\tfrac{S}{S^*})+(Q_t-Q_t^*-Q_t^*\log\tfrac{Q_t}{Q_t^*}) \\ & +(Q_p-Q_p^*-Q_p^*\log\tfrac{Q_p}{Q_p^*}) \end{aligned} \tag{35}$$

Using the derivative on both sides, we obtain

$$\frac{dL}{dt} = L = (\frac{P-P^*}{P})\dot{P} + (\frac{O-O^*}{O})\dot{O} + (\frac{S-S^*}{S})\dot{S}$$
$$(\frac{Q_t-Q_t^*}{Q_t})\dot{Q}_t + (\frac{Q_p-Q_p^*}{Q_p})\dot{Q}_p \tag{36}$$

we have

$$\begin{aligned}\frac{dL}{dt} &= (\frac{P-P^*}{P})(\mu - \alpha PS - \nu P)\\ &+ (\frac{O-O^*}{O})(\alpha PS - \beta_1 O - \nu O)\\ &+ (\frac{S-S^*}{S})(\beta_1 O + \beta_2 S Q_t - (\nu+\delta+\varepsilon)S)\\ &+ (\frac{Q_t-Q_t^*}{Q_t})(-\beta_2 S Q_t + \delta(1-\gamma)S - \nu Q_t)\\ &+ (\frac{Q_p-Q_p^*}{Q_p})(\gamma\delta S - \nu Q_p)\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{dL}{dt} &= (\frac{P-P^*}{P})(\mu - \alpha(P-P^*)(S-S^*) - \nu(P-P^*)) + (\frac{O-O^*}{O})\\ &\times (\alpha(P-P^*)(S-S^*) - \beta_1(O-O^*) - \nu(O-O^*)) + (\frac{S-S^*}{S})\end{aligned}$$

$$\begin{aligned}&\times (\beta_1(O-O^*) + \beta_2(S-S^*)(Q_t-Q_t^*) - (\nu+\delta+\varepsilon)(S-S^*))\\ &+ (\frac{Q_t-Q_t^*}{Q_t})(-\beta_2(S-S^*)(Q_t-Q_t^*) + \delta(1-\gamma)(S-S^*)\\ &- \nu(Q_t-Q_t^*)) + (\frac{Q_p-Q_p^*}{Q_p})(\gamma\delta(S-S^*) - \nu(Q_p-Q_p^*))\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{dL}{dt} &= \mu - \mu\frac{P^*}{P} - \alpha\frac{S}{P}(P-P^*)^2 + \alpha\frac{S^*}{P}(P-P^*) - \nu\frac{(P-P^*)^2}{P}\\ &+ \alpha PS - \alpha PS^* - \alpha SP^* + \alpha P^*S^* - \alpha\frac{O^*}{O}PS + \frac{O^*}{O}P^*S\\ &+ \frac{O^*}{O}P^* - \frac{O^*}{O}P^*S^* - \beta_1\frac{(O-O^*)^2}{O} - \nu\frac{(O-O^*)^2}{O}\\ &+ \beta_1 O - \beta_1 O^* - \frac{S^*}{S}\beta_1 O + \frac{S^*}{S}\beta_1 O^* + \frac{(S-S^*)^2}{S}Q_t\beta_2\\ &- \frac{(S-S^*)^2}{S}Q_t^*\beta_2 - (\nu+\delta+\varepsilon)\frac{(S-S^*)^2}{S} - \beta_2 S\frac{(Q_t-Q_t^*)^2}{Q_t}\\ &+ \beta_2 S^*\frac{(Q_t-Q_t^*)^2}{Q_t} + \delta S - \gamma\delta S - \delta S^* - \gamma\delta S^* - \delta S\frac{Q_t^*}{Q_t}\\ &+ \gamma\delta S\frac{Q_t^*}{Q_t} + \delta S^*\frac{Q_t^*}{Q_t} - \gamma\delta S^*\frac{Q_t^*}{Q_t} - \nu\frac{(Q_t-Q_t^*)^2}{Q_t}\gamma\delta S\\ &- \gamma\delta S^* - \frac{Q_p^*}{Q_p}\gamma\delta S\frac{Q_p^*}{Q_p}\gamma\delta S^* - \nu(\frac{(Q_p-Q_p^*)^2}{Q_p}\end{aligned}$$

we can write

$$\frac{dL}{dt} = j + l \tag{39}$$

where

$$\begin{aligned} j &= \mu + \alpha\frac{S^*}{P}(P-P^*) + \alpha PS + \alpha P^*S^* + \frac{O^*}{O}P^*S \\ &+ \frac{O^*}{O}P^* + \beta_1 O + \frac{S^*}{S}\beta_1 O^* + \frac{(S-S^*)^2}{S}Q_t\beta_2 + \gamma\delta S\frac{Q_t^*}{Q_t} \\ &+ \gamma\delta S\frac{Q_t^*}{Q_t} + \delta S^*\frac{Q_t^*}{Q_t} + \gamma\delta S\frac{Q_t^*}{Q_t} + \delta S^*\frac{Q_t^*}{Q_t}\gamma\delta S + \frac{Q_p^*}{Q_p}\gamma\delta S^* \end{aligned} \tag{40}$$

and

$$\begin{aligned} l &= \mu\frac{P^*}{P} + \alpha\frac{S}{P}(P-P^*)^2 + \nu\frac{(P-P^*)^2}{P} + \alpha PS^* + \alpha SP^* \\ &+ \alpha\frac{O^*}{O}PS + \frac{O^*}{O}P^*S^* + \beta_1\frac{(O-O^*)^2}{O} + \nu\frac{(O-O^*)^2}{O} \\ &+ \beta_1 O^* + \frac{S^*}{S}\beta_1 O + \frac{(S-S^*)^2}{S}Q_t^*\beta_2 + (\nu+\delta+\varepsilon)\frac{(S-S^*)^2}{S} \\ &+ \beta_2 S\frac{(Q_t-Q_t^*)^2}{Q_t} + \gamma\delta S - \delta S^* + \gamma\delta S^* + \delta S\frac{Q_t^*}{Q_t} \\ &+ \gamma\delta S^*\frac{Q_t^*}{Q_t} + \nu\frac{(Q_t-Q_t^*)^2}{Q_t} + \gamma\delta S^* + \frac{Q_p^*}{Q_p}\gamma\delta S + \nu(\frac{(Q_p-Q_p^*)^2}{Q_p} \end{aligned} \tag{41}$$

We conclude that if $j < l$, this yields $\frac{dL}{dt} = 0$. However, when $P = P^*, O = O^*, S = S^*, Q_t = Q_t^*, Q_p = Q_p^*$,

$$0 = j - l \Rightarrow \frac{dL}{dt} = 0 \tag{42}$$

we can see that

$$(P^*, O^*, S^*, Q_t^*, Q_p^*) \in T\frac{dL}{dt} = 0 \tag{43}$$

According to LaSalle's invariance principle, the model is globally asymptotically stable.

6. Numerical Scheme by FFM.

In this section, we present a numerical scheme to solve the model numerically, based on a Newton polynomial. Here, we apply novel differential and integral operators to the proposed model. In this case, the operator with the Mittag-Leffler kernel will replace the classical differential operator. The version with a variable order will also be used.

$$\begin{aligned} 0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= \mu - \alpha P(\eta)S(\eta) - \nu P(\eta) \\ 0^{FFM}D_t^{\zeta,\tau}O'(\eta) &= \alpha P(\eta)S(\eta) - \beta_1 O(\eta) - \nu O(\eta) \\ 0^{FFM}D_t^{\zeta,\tau}S'(\eta) &= \beta_1 O(\eta) + \beta_2 S(\eta)Q_t(\eta) - (\nu+\delta+\varepsilon) \end{aligned}$$

$$0^{FFM}D_t^{\zeta,\tau}Q_t'(\eta) \quad = \quad -\beta_2 S(\eta)Q_t(\eta)+\delta(1-\gamma)S(\eta)-\nu Q_t(\eta)$$

$$0^{FFM}D_t^{\zeta,\tau}Q_p'(\eta) \quad = \quad \gamma\delta S(\eta)-\nu Q_p(\eta)$$

To keep things simple, we write the above equation as;

$$\begin{aligned}
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= P_1(t,P,O,S,Q_t,Q_p) \\
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= O_1(t,P,O,S,Q_t,Q_p) \\
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= S_1(t,P,O,S,Q_t,Q_p) \qquad (44) \\
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= Q_{t1}(t,P,O,S,Q_t,Q_p) \\
0^{FFM}D_t^{\zeta,\tau}P'(\eta) &= Q_{p1}(t,P,O,S,Q_t,Q_p)
\end{aligned}$$

Where

$$\begin{aligned}
P_1(t,P,O,S,Q_t,Q_p) &= \mu-\alpha P(\eta)S(\eta)-\nu P(\eta) \\
O_1(t,P,O,S,Q_t,Q_p) &= \alpha P(\eta)S(\eta)-\beta_1 O(\eta)-\nu O(\eta) \\
S_1(t,P,O,S,Q_t,Q_p) &= \beta_1 O(\eta)+\beta_2 S(\eta)Q_t(\eta)-(\nu+\delta+\varepsilon) \\
Q_{t1}(t,P,O,S,Q_t,Q_p) &= -\beta_2 S(\eta)Q_t(\eta)+\delta(1-\gamma)S(\eta)-\nu Q_t(\eta) \\
Q_{p1}(t,P,O,S,Q_t,Q_p) &= \gamma\delta S(\eta)-\nu Q_p(\eta)
\end{aligned}$$

We acquire the subsequent

$$P(t_{\sigma+1})=\frac{\tau(1-\zeta)}{AB(\zeta)}t_\sigma^{\tau-1}P_1(t_\sigma,P(t_\sigma),O(t_\sigma),S(t_\sigma),Q_t(t_\sigma),Q_t(t_\sigma))$$
$$\frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)}\Sigma_{\nu=2}^{\sigma}\int_{t_\nu}^{t_{\nu+1}}P_1(t,P,O,S,Q_t,Q_p)\rho^{\tau-1}(t_{\sigma+1}-\rho)^{\zeta-1}d\rho$$

$$O(t_{\sigma+1})=\frac{\tau(1-\zeta)}{AB(\zeta)}t_\sigma^{\tau-1}O_1(t_\sigma,P(t_\sigma),O(t_\sigma),S(t_\sigma),Q_t(t_\sigma),Q_t(t_\sigma))$$
$$\frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)}\Sigma_{\nu=2}^{\sigma}\int_{t_\nu}^{t_{\nu+1}}O_1(t,P,O,S,Q_t,Q_p)\rho^{\tau-1}(t_{\sigma+1}-\rho)^{\zeta-1}d\rho$$

$$S(t_{\sigma+1})=\frac{\tau(1-\zeta)}{AB(\zeta)}t_\sigma^{\tau-1}S_1(t_\sigma,P(t_\sigma),O(t_\sigma),S(t_\sigma),Q_t(t_\sigma),Q_t(t_\sigma))$$
$$\frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)}\Sigma_{\nu=2}^{\sigma}\int_{t_\nu}^{t_{\nu+1}}S_1(t,P,O,S,Q_t,Q_p)\rho^{\tau-1}(t_{\sigma+1}-\rho)^{\zeta-1}d\rho$$

$$Q_t(t_{\sigma+1})=\frac{\tau(1-\zeta)}{AB(\zeta)}t_\sigma^{\tau-1}Q_{t1}(t_\sigma,P(t_\sigma),O(t_\sigma),S(t_\sigma),Q_t(t_\sigma),Q_t(t_\sigma))$$
$$\frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)}\Sigma_{\nu=2}^{\sigma}\int_{t_\nu}^{t_{\nu+1}}Q_{t1}(t,P,O,S,Q_t,Q_p)\rho^{\tau-1}(t_{\sigma+1}-\rho)^{\zeta-1}d\rho$$

$$Q_p(t_{\sigma+1}) = \frac{\tau(1-\zeta)}{AB(\zeta)} t_\sigma^{\tau-1} Q_{p1}(t_\sigma, P(t_\sigma), O(t_\sigma), S(t_\sigma), Q_t(t_\sigma), Q_t(t_\sigma))$$
$$\frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)} \Sigma_{v=2}^{\sigma} \int_{t_v}^{t_{v+1}} Q_{p1}(t, P, O, S, Q_t, Q_p) \rho^{\tau-1} (t_{\sigma+1}-\rho)^{\zeta-1} d\rho$$

The Newton polynomial, which we will review, is provided by

$$\begin{aligned}
G(t,P,O,S,Q_t,Q_p) &\simeq G(t_{\sigma-2}, P_{\sigma-2}, O_{\sigma-2}, S_{\sigma-2}, Q_{t(\sigma-2)}, Q_{p(\sigma-2)}) \\
&+ \frac{1}{\Delta t}[G(t_{\sigma-1}, P_{\sigma-1}, O_{\sigma-1}, S_{\sigma-1}, Q_{t(\sigma-1)}, Q_{p(\sigma-1)}) \\
&- G(t_{\sigma-2}, P_{\sigma-2}, O_{\sigma-2}, S_{\sigma-2}, Q_{t(\sigma-2)}, Q_{p(\sigma-2)})](\rho - t_{\sigma-2}) \\
&+ \frac{1}{2\Delta t^2}[G(t_\sigma, P_\sigma, O_\sigma, S_\sigma, Q_{t(\sigma)}, Q_{p(\sigma)}) \\
&- 2G(t_{\sigma-1}, P_{\sigma-1}, O_{\sigma-1}, S_{\sigma-1}, Q_{t(\sigma-1)}, Q_{p(\sigma-1)}) \\
&- G(t_{\sigma-2}, P_{\sigma-2}, O_{\sigma-2}, S_{\sigma-2}, Q_{t(\sigma-2)}, Q_{p(\sigma-2)})] \\
&\times (\rho - t_{\sigma-2})(\rho - t_{\sigma-1})
\end{aligned}$$

Here are the numerical solutions for $P(\eta)$ when the Newton polynomial is substituted into the equations.

$$\begin{aligned}
P^{(\sigma+1)} &= \frac{\tau(1-\zeta)}{AB(\zeta)} t_\sigma^{\tau-1} P_1(t_\sigma, P(t_\sigma), O(t_\sigma), S(t_\sigma), Q_t(t_\sigma), Q_t(t_\sigma)) \\
&+ \frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)} \Sigma_{v=2}^{\sigma} P_1(t_{v-2}, P^{v-2}, S^{v-2}, O^{v-2}, Q_t^{v-2}, Q_p^{v-2}) t_{v-2}^{\tau-1} \\
&\times \int_{t_v}^{t_{v+1}} (t_{\sigma+1}-\rho)^{\zeta-1} d\rho + \frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)} \Sigma_{v=2}^{\sigma} \frac{1}{\triangle t}[t_{v-1}^{\tau-1} \\
&\times P_1(t_{v-1}, P^{v-1}, O^{v-1}, S^{v-1}, Q_t^{v-1}, Q_p^{v-1}) \\
&- t_{v-2}^{\tau-1} P_1(t_{v-2}, P^{v-2}, S^{v-2}, O^{v-2}, Q_t^{v-2}, Q_p^{v-2})] \\
&\times \int_{t_v}^{t_{v+1}} (\rho - t_{v-2})(t_{\sigma+1}-\rho)^{\zeta-1} d\rho
\end{aligned}$$

$$\begin{aligned}
&+ \frac{\zeta\tau}{AB(\zeta)\Gamma(\zeta)} \Sigma_{v=2}^{\sigma} \frac{1}{2\triangle t^2} t_v^{\tau-1} P_1(t_v, P^v, O^v, S^v, Q_t^v, Q_p^v) \\
&- 2t_{v-1}^{\tau-1} P_1(t_{v-1}, P^{v-1}, O^{v-1}, S^{v-1}, Q_t^{v-1}, Q_p^{v-1}) \\
&+ t_{v-2}^{\tau-1} P_1(t_{v-2}, P^{v-2}, O^{v-2}, S^{v-2}, Q_t^{v-2}, Q_p^{v-2})
\end{aligned}$$

For the integral in the aforementioned equation, we can carry out the following calculations.

$$\begin{aligned}
\int_{t_v}^{t_{v+1}} (t_{\sigma+1}-\rho)^{\zeta-1} d\rho &= \frac{(\Delta t)^\zeta}{\zeta}[(\sigma - v + 1)^\zeta - (\sigma - v)^\zeta] \\
\int_{t_v}^{t_{v+1}} (\rho - t_{v-2})(t_{\sigma+1}-\rho)^{\zeta-1} d\rho &= \frac{(\Delta t)^{\zeta+1}}{\zeta(\zeta+1)}
\end{aligned}$$

$$\int_{t_v}^{t_{v+1}} (\rho - t_{v-2}(\rho - t_{v-1})(t_{\sigma+1}-\rho)^{\zeta-1} d\rho = \frac{(\Delta t)^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \times [(\sigma - v + 1)^\zeta \times \{2(\sigma - v)^2$$
$$+(3\zeta+10)(\sigma - v) + 2\zeta^2 + 9\zeta + 12\} - (\sigma - v)^\zeta \times \{2(\sigma - v)^2 + (5\zeta+10)(\sigma - v) + 6\zeta^2 + 18\zeta + 12\}] \quad (45)$$

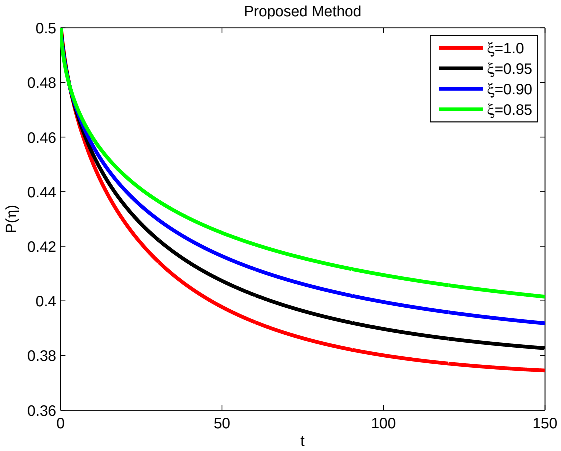
$$
\begin{aligned}
P^{(\sigma+1)} &= \frac{\tau(1-\zeta)}{AB(\zeta)} t_{\sigma}^{\tau-1} P_1(t_{\sigma}, P(t_{\sigma}), O(t_{\sigma}), S(t_{\sigma}), Q_t(t_{\sigma}), Q_t(t_{\sigma})) + \frac{\zeta(\Delta t)^{\zeta}}{AB(\zeta)\Gamma(\zeta+1)} \\
&\Sigma_{v=2}^{\sigma} t_{v-2}^{\tau-1} P_1(t_{v-2}, P^{v-2}, S^{v-2}, O^{v-2}, Q_t^{v-2}, Q_p^{v-2}) \times [(\sigma - v + 1) - (\sigma - v)] \\
&+ \frac{\zeta(\Delta t)^{\zeta}}{AB(\zeta)\Gamma(\zeta+2)} \Sigma_{v=2}^{\sigma} [t_{v-1}^{\tau-1} P_1(t_{v-1}, P^{v-1}, O^{v-1}, S^{v-1}, Q_t^{v-1}, Q_p^{v-1}) - t_{v-1}^{\tau-1} \\
&P_1(t_{v-2}, P^{v-2}, S^{v-2}, O^{v-2}, Q_t^{v-2}, Q_p^{v-2})] \times [(\sigma + v + 1)\zeta(\sigma - v + 3 + 2\zeta) \\
&- (\sigma - v)^{\zeta}(\sigma - v + 3 + 3\zeta)] + \frac{\zeta(\Delta t)^{\zeta}}{2AB(\zeta)\Gamma(\zeta+3)} \Sigma_{v=2}^{\sigma} [t_v^{\tau-1} P_1(t_v, P^v, O^v, S^v, Q_t^v, Q_p^v) \\
&- 2t_{v-1}^{\tau-1} P_1(t_{v-1}, P^{v-1}, O^{v-1}, S^{v-1}, Q_t^{v-1}, Q_p^{v-1}) + t_{v-2}^{\tau-1} P_1(t_{v-2}, P^{v-2}, O^{v-2}, S^{v-2}, Q_t^{v-2}, Q_p^{v-2})] \\
&\times [(\sigma - v + 1)^{\zeta} \{2(\sigma - v)^2 + (3\zeta + 10)(\sigma - v) + 2\zeta^2 + 9\zeta + 12\} \\
&- (\sigma - v)^{\zeta} \{2(\sigma - v)^2 + (5\zeta + 10)(\sigma - v) + 6\zeta^2 + 18\zeta^2 + 12\}]
\end{aligned}
$$

The numerical scheme for $O(\eta)$, $S(\eta)$, $Q_t(\eta)$ and $Q_p(\eta)$ are similar to all above equations so its omit. This is the complete numerical scheme of fractal fractional with ML kernel.

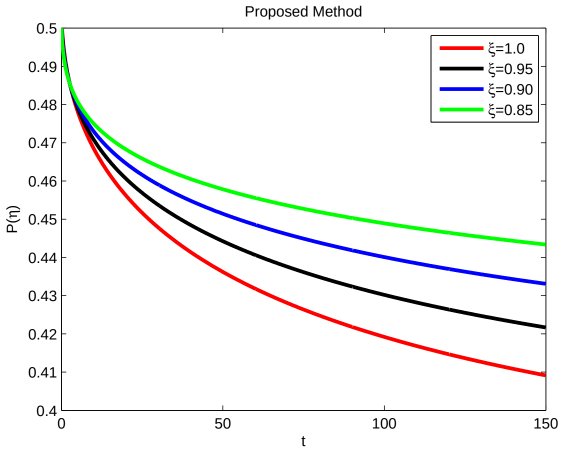
7. Simulation

Here, We employed a sophisticated method to calculate theoretical outcomes and assess their relevance. The validity of the theoretical results is demonstrated by the following examples. By means of simulation, the suggested POSQ_tQ_p system is described in terms of actual circumstances. Applying non-integer parametric values to the smoking model's chronic stage produced some amazing results. The answers for $P(\eta)$, $O(\eta)$, $S(\eta)$, $Q_t(\eta)$ and $Q_p(\eta)$ are shown in Figures 1-5. To confirm that the theoretical results are effective, we provide the following examples with MATLAB coding. The smoking model was simulated numerically. The recommended system employs the following parameters: $\alpha = 0.14$, $\mu = 0.001$, $\nu = 0.001$, $\beta_1 = 0.002$, $\beta_2 = 0.0025$, $\delta = 0.08$, $\varepsilon = 0.00003$, $1-\gamma = 0.52$ $\gamma = 0.48$

Potential somkers $P(\eta)$, occasional smokers $O(\eta)$, smokers $S(\eta)$, temporary quitters Q_t, and permanent quitters Q_p in which all sub-compartments shrink and eventually stabilize using various dimensions, as indicated by figures 4 and 5 respectively. Comparable actions are seen either with dimension 0.7 or 0.5 with slight impacts, but by decreasing dimensions we obtain more suitable results as shown in figure respectively. The behavior of the dynamics with in the special fractional parameters is displayed by the numerical results that have been provided. Additionally, it is noted that recovered increases by decreasing the dimension and fractional values as shown in Figure under the acute and chronic stages. It is concluded that early detection of smoking and a combination of acute and chronic stage investigation can help control smoking.

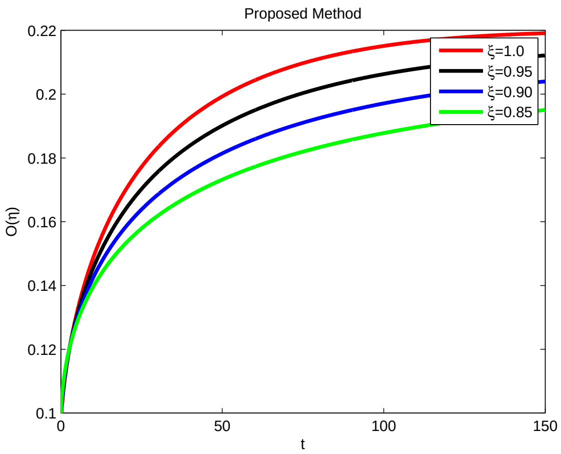


(a) proportion 0.7

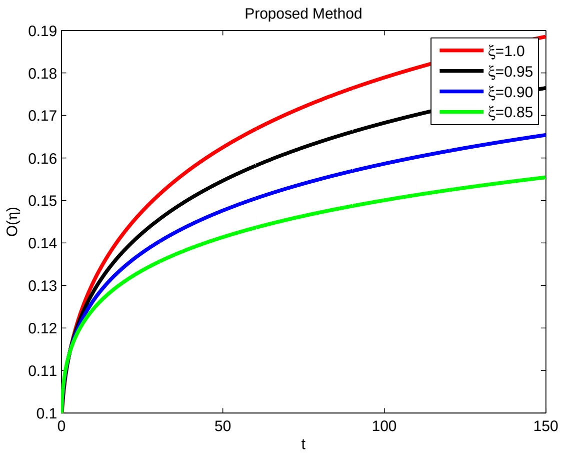


(b) proportion 0.5

Figure 1: Graphical Representation of $P(\eta)$.

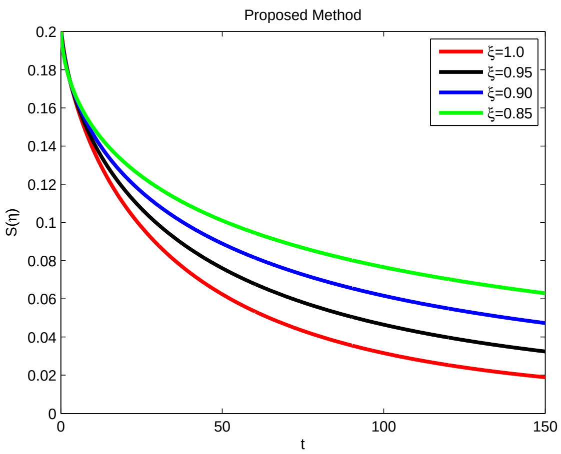


(a) proportion 0.7

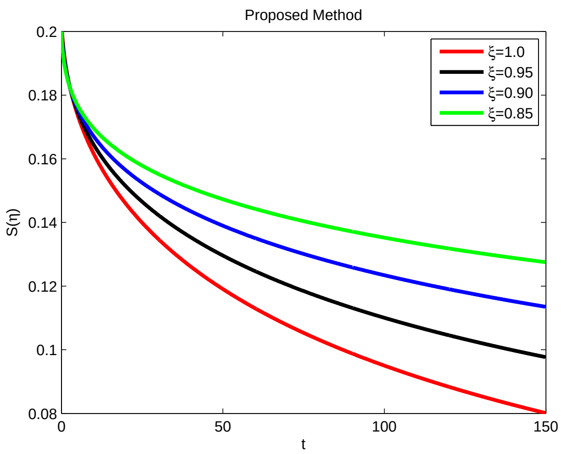


(b) proportion 0.5

Figure 2: Graphical Representation of $Q(\eta)$.

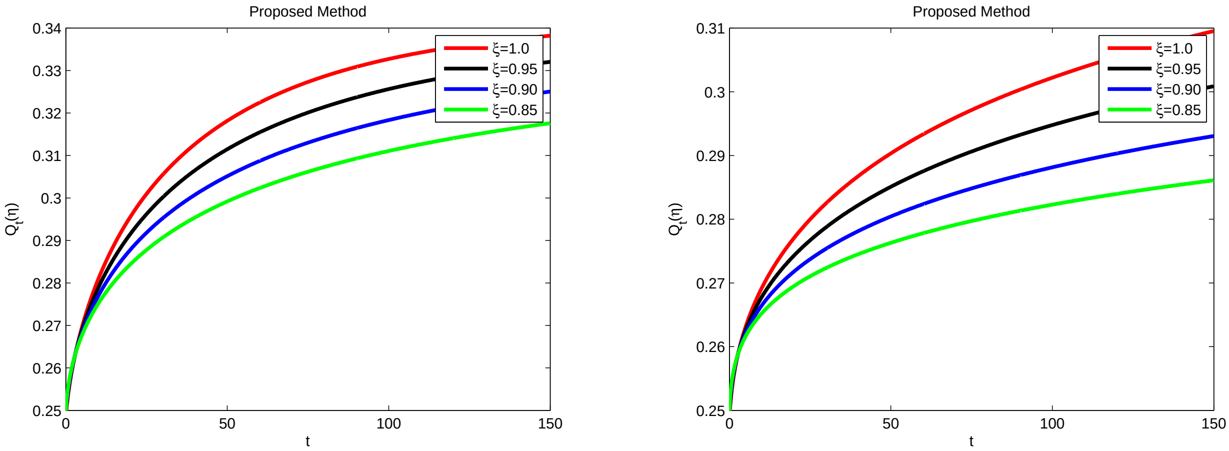


(a) proportion 0.7



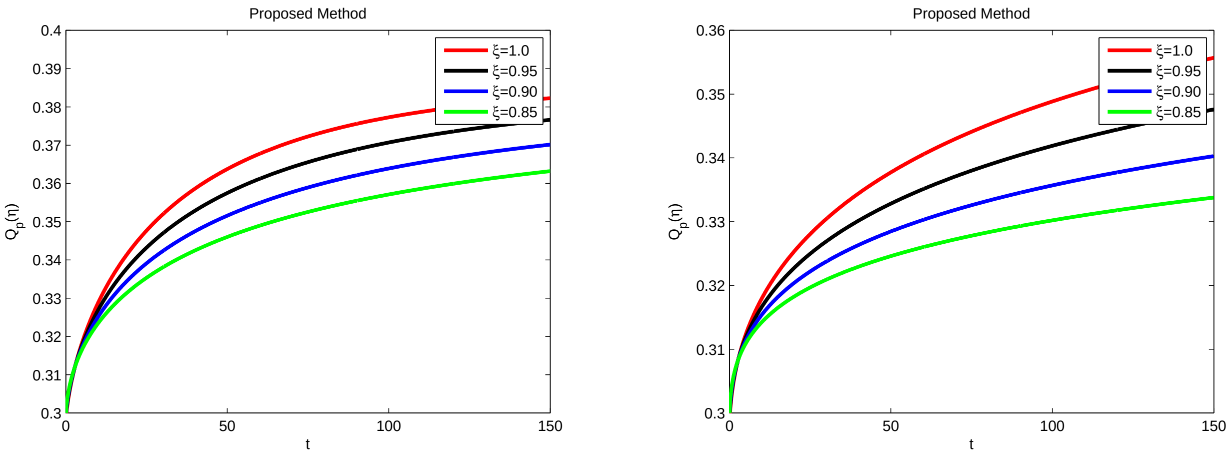
(b) proportion 0.5

Figure 3: Graphical Representation of $S(\eta)$.



(a) proportion 0.7 **(b)** proportion 0.5

Figure 4: Graphical Representation of $Qt(\eta)$.



(a) proportion 0.7 **(b)** proportion 0.5

Figure 5: Graphical Representation of $Qp(\eta)$.

8. Conclusion

In this paper, investigation and analysis of smoking model has been done by utilizing the fractal fractional operator for continuous monitoring of bad impact of smoking. Qualitative and quantitative analysis have been made for the proposed system to capture the stable state of the smoking system. The boundedness, uniqueness and positivity of the smoking model is derived which are the key properties of the epidemic system. System is investigated globally by utilizing lypounove function to observe the rate of impact. Numerical solutions are taken out under advance operator FFM for continuous monitoring of bad impact of smoking with different dimensions. Simulations are also derived with the help of MATLAB to capture the real behavior of bad impact of smoking. Furthermore, based on our findings, we provide future estimates to help reduce the risk of disease transmission in the environment.

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References:

[1] C. S. Chou and A. Friedman,Introduction,in Introduction to Mathematical Biology. Springer Undergraduate Texts in Mathematics and Technology, Springer, Cham, 2016.

[2] Biazar J (2006). Solution of the epidemic model by Adomian decomposition method. Applied Mathematics and Computation, 173(2): 1101-1106.

[3] Makinde OD (2007). Adomian decomposition approach to a SIR epidemic model with constant vaccination strategy. Applvcied Mathematics and Computation, 184(2): 842-848.

[4] Arafa AAM, Rida SZ, and Khalil M (2012). Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection. Nonlinear Biomedical Physics, 6(1): 1-7.

[5] Erturk VS, Zaman G, and Momani S (2012). A numericanalytic method for approximating a giving up smoking model containing fractional derivatives. Computers and Mathematics with Applications, 64(10): 3065-3074.

[6] Centers for Disease Control and Prevention (CDC), Annual smoking attributable mortality, years of potential life lost, and economic cost-united state 1995-1999, MMWR. Morbidity and Mortality Weekly Report, vol. 51, no. 14, pp. 300303, 2002.

[7] A. Jemal, M. M. Center, C. Desantis, and E. M. Ward, Global patterns of cancer incidence and mortality rates and trends, Cancer Epidemiology, Biomarkers & Prevention, vol. 19, pp. 18931907, 2010.

[8] A. Lahrouz, L. Omari, D. Kiouach, A. Belmaati, Deterministic and stochastic stability of a mathematical model of smoking, Statist. Probab. Lett. 81 (8) (2011) 12761284.

[9] World Health Organization: WHO. (2022). Tobacco. www.who. int. https://www.who.int/newsroom/fact-sheets/detail/tobacco.

[10] H. Singh, D. Baleanu, J. Singh, H. Dutta, Computational study of fractional order smoking model, Chaos Solitons Fractals 142 (2021) 110440.

[11] S. Ucar, E. Ucar, N. Ozdemir, Z. Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana-Baleanu derivative, Chaos Solitons Fractals 118 (2019) 300306

[12] Z.A. Khan, M.U. Rahman, K. Shah, Study of a fractal fractional smoking models with relapse and harmonic mean type incidence rate, J. Function Spaces (2021) 111.

[13] B. Melkamu, B. Mebrate, A fractional model for the dynamics of smoking tobacco using caputo-fabrizio derivative, J. Appl. Math. (2022) 110, https://doi.org/10.1155/2022/2009910.

[14] Ahmad, Aqeel, et al. ”Flip bifurcation analysis and mathematical modeling of cholera disease by taking control measures.” Scientific Reports 14.1 (2024): 10927.

[15] Ahmad, Aqeel, et al. ”Stability Analysis of SARSCoV2 with Heart Attack Effected Patients and Bifurcation.” Advanced Biology 8.4 (2024): 2300540.

[16] H. Bulut, H. M. Baskonus, and F. B. M. Belgecam, The analytic solutions of some fractional ordinary differential equation by Sumudu transform method, Abstr. Appl. Anal., vol. 2013, article 203875, 6 pages, 2013.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[18] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, Chaos, Solitons and Fractals, vol. 89, pp. 447454, 2016.

[19] L. L. Huang, D. Baleanue, and S. D. Wu Zeng, A new application of the fractional logistic map, Romanian Journal of Physics, vol. 61, pp. 11721179, 2016.

[20] G. Zaman, Qualitative behavior of giving up smoking model, Bulletin of the Malaysian Mathematical Sciences Society, vol. 34, pp. 403415, 2011.

[21] J. Singh, D. Kumar, M. A. Qurashi, and D. Baleanu, A new fractional model for giving up smoking dynamics, Advances in Difference Equations, vol. 2017, no. 1, Article ID 88, 2017.

[22] Arafa AAM, Rida SZ, and Khalil M (2012). Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection. Nonlinear Biomedical Physics, 6(1): 1-7.

[23] Zaman G (2011a). Qualitative behavior of giving up smoking models. Bulletin of the Malaysian Mathematical Sciences Society, 34(2): 403-415.

[24] Lubin JH and Caporaso NE (2006). Cigarette smoking and lung cancer: modeling total exposure and intensity. Cancer Epidemiology and Prevention Biomarkers, 15(3): 517-523.

[25] Garsow CC, Salivia GJ, and Herrera AR (2000). Mathematical models for the dynamics of tobacoo use, recovery and relapse. Technical Report Series BU-1505-M, Cornell University, UK.

[26] Mickens RE (1989). Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis. Numerical Methods for Partial Differential Equations, 5(4): 313-325.

[27] Sharomi O and Gumel AB (2008). Curtailing smoking dynamics: a mathematical modeling approach. Applied Mathematics and Computation, 195(2): 475-499.

[28] Zeb A, Chohan MI, and Zaman G (2012). The homotopy analysis method for approximating of giving up smoking model in fractional order. Applied Mathematics, 3(8): 914-919.

[29] Alkhudhari Z, Al-Sheikh S, and Al-Tuwairqi S (2014). Global dynamics of a mathematical model on smoking. Hindawi Publishing Corporation, ISRN Applied Mathematics, 2014: Article ID 847075, 7 pages. http://doi.org/10.1155/2014/ 847075

[30] C. Castillo-Garsow, G. Jordan-Salivia, A.E. Rodriguez-Herrera, Mathematical models for the dynamics of tobacco use, Recovery and Relapse. (1997). http://ecommons.cornell.edu/bitstream/1813/32095/1/BU-1505-M.pdf.

[31] O. Sharomi, A.B. Gumel, Curtailing smoking dynamics: a mathematical modeling approach, Appl. Math Comput. 195 (2) (2008) 475499.

[32] G. Zaman, Qualitative behavior of giving up smoking models, Bulletin of the Malaysian Mathematical Sciences Soc. Second Series 34 (2) (2011) 403415.

[33] R. Ullah, M. Khan, G. Zaman, S. Islam, M.A. Khan, S. Jan, T. Gul, Dynamical features of a mathematical model on smoking, J. Appl. Environ. Biol. Sci 6 (1) (2016) 9296.

[34] Ahmad, Aqeel, et al. ”Analytical analysis and bifurcation of pine wilt dynamical transmission with host vector and nonlinear incidence using sustainable fractional approach.” Partial Differential Equations in Applied Mathematics 11 (2024): 100830.

[35] Khurram Faiz, Aqeel Ahmad, Muhammad Suleman Khan, Safdar Abbas, Control of Marburg Virus Disease Spread in Humans under Hypersensitive Response through Fractal-Fractional . 2024. Journal of Mathematical Modeling and Fractional Calculus 1 (1): 88-113

[36] P. Xiao, Z. Zhang, X. Sun, Smoking dynamics with health education effect, AIMS Mathematics 3 (4) (2018) 584599.

[37] B. Fekede, B. Mebrate, Sensitivity and mathematical model analysis on secondhand smoking tobacco, J. Egyptian Math. Soc. 28 (1) (2020) 50.

[38] C. Liu, W. Sun, X. Yi, Optimal control of a fractional smoking system, J. Industrial and Manage. Optimization 19 (4) (2023) 29362954.

[39] P. Veeresha, D.G. Prakasha, H.M. Baskonus, Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method, Mathematical Sciences 13 (2019) 115128.

[40] P. Veeresha, N.S. Malagi, D.G. Prakasha, H.M. Baskonus, An efficient technique to analyze the fractional model of vectorborne diseases, Phys. Scr. 97 (5) (2022) 054004.

[41] P. Veeresha, E. Ilhan, D.G. Prakasha, H.M. Baskonus, W. Gao, Regarding on the fractional mathematical model of Tumour invasion and metastasis, Comput. Model. Eng. Sci. 127 (3) (2021) 10131036.

[42] C. Maji, F. Al Basir, D. Mukherjee, C. Ravichandran, K. Nisar, COVID-19 propagation and the usefulness of awareness-based control measures: a mathematical model with delay, AIMS Math 7 (7) (2022) 1209112105

[43] Ahmad, Aqeel, et al. "Study on symptomatic and asymptomatic transmissions of COVID-19 including flip bifurcation." International Journal of Biomathematics (2024): 2450002.

[44] Z. Zhang, W. Zhang, K.S. Nisar, N. Gul, A. Zeb, V. Vijayakumar, Dynamical aspects of a tuberculosis transmission model incorporating vaccination and time delay, Alex. Eng. J. 66 (2023) 287300.

[45] V. Vijayakumar, K.S. Nisar, A. Shukla, B. Hazarika, R. Samidurai, An investigation on the approximate controllability of impulsive neutral delay differential inclusions of second order, Mathematical Methods in the Appl. Sciences (2022).

[46] K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, Asian J. Control 24 (3) (2022) 14061415.