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Analysis of Energy Sector Model with Laplace Transform under modified fractional operator: Mathematical Analysis

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Abstract: Population growth is driving up energy demand, and burning fossil fuels accounts for a sizable amount of global energy production. Since this adds to the rise in greenhouse gas emissions, lowering energy sector emissions is essential to achieving climate change goals. In this research, three hybrid fractional operators are used to investigate carbon emissions from the power sector. A hybrid fractional operator is used to modify differential equations throughout time and space in order to create a mathematical model that describes the human population, energy consumption, and atmospheric carbon dioxide concentration. Compartmental models offer a mathematical framework for comprehending systems and forecasting outcomes, which makes them essential fgor comprehending real-world events. The effectiveness of these novel operators is supported by a number of analyses, and the Lipschitz condition ensures unique results. The order of the fractional derivative has a major impact on the dynamical process that is used to construct the non-integer order model.

Keywords: Carbon dioxide emission; Mathematical Model; Fractional operators; Lipschitz criteria;

1. Introduction

The socioeconomic advancement of a country is greatly influenced by the energy sector, which is growing due to population and economic expansion. It is estimated that between 2010 and 2040, worldwide energy consumption would increase by 56% [1]. Issues about the rising atmospheric concentration and its effect on climate change have prompted international research into recording and sustaining CO2 emissions from small-scale sources, especially power plants that produce electricity from fossil fuels like coal, oil, and natural gas [2]. With the previous two decades accounting for 55% of the greenhouse effect, human activity has caused the concentration of CO2 in the atmosphere to rise by 25% after the industrial period. Burning fossil fuels is the most significant cause of greenhouse gas emissions worldwide, accounting for 60% of emissions, especially in the northern hemisphere [3]. There are several ways to mitigate carbon dioxide emissions, but renewable energy is the best option because it produces fewer emissions over the course of its life cycle than fossil fuels. Additionally, nuclear energy is an economical choice. In order to combat global warming, the conversation highlights the necessity of a comprehensive study of the dynamics of carbon dioxide in the presence of plants and mining operations [5]. For a long time, the spread of carbon dioxide has been forecast via mathematical models. If the underlying assumptions they make about real-world systems are correct, they can be trusted. Mathematical models can help researchers make better plans and decisions. To investigate the effects of different factors on the concentration of carbon dioxide in the atmosphere, various mathematical models [6]-[10] have been proposed.

Fractional calculus employs differentiation and integration with fractional order due to its memory and genetic properties, which makes it more effective than standard integer order for modeling phenomena and elucidating real-world issues. Fractional differential equations are crucial for representing real-world scenarios in many fields [11]-[25]. Taking into account both theoretical and numerical elements, Khan et al. [26] estimated the impact of fast GHG emissions on coastal ecosystems and climatic changes using a fractal-fractional hybrid model. Using the Caputo fractional operator, a fractional-order model was created in [27] to examine the effects of increasing global warming on aquatic ecosystems while taking the environment and organisms throughout time into account. Taking into account the interaction between oxygen generation, humidity, and duration, Premakumari et al. [28] created a model utilizing the Caputo fractional derivative to examine how climate change affects the dynamics of oxygen, microorganisms, and fish. Some relevant research is provided in [29, 30]. Baleanu et al. [32] suggested a more versatile and generalized operator, the constant-proportional Caputo fractional derivative. Furthermore Ali Akgül [33] coupled the proportional derivative with two well-known fractional derivatives, yielding several useful results based on these definitions as can be seen in [34]-[40].

In order to propose a new mathematical structure, this study uses recently found hybrid fractional operators to create a fractional order model of carbon dioxide emissions from the energy sector. The following is the manuscript's structure:

- In Section 1, the literature is reviewed and an introduction is given.
- We go over the basics of the fractional operator utilized in the suggested model in Section 2.
- In Section 3, we introduce a hybrid fractional operator fractional order model for energy sector carbon dioxide emissions.
- The proposed model's in-depth qualitative analysis is covered in Section 4.
- More analysis is provided in Section 5 using hybrid fractional operators that have just been constructed.
- Using the Laplace-Adomian decomposition approach, the model's analytical solution is determined in section 6.
- The results and significant conclusions of our analysis are described in Section 7.

2. Key Concepts

For managing mathematical models and comprehending the dynamics of complicated systems, fractional calculus is essential. Performance is enhanced by its adjustable time and frequency responses. Important ideas in fractional calculus for our system analysis are given below.

Definition 2.1. *[31] The Caputo derivative of $\Phi(t)$ is defined by*

$$ {}_{0}^{C}\mathrm{D}_{t}^{\varphi}\Phi(t)=\frac{1}{\Gamma(1-\varphi)}\int_{0}^{t}\Phi'(\rho)(t-\rho)^{-\varphi}d\rho. \tag{1}$$

Definition 2.2. *[31] The Riemann-Liouville (RL) integral is defined by*

$$ {}_{0}^{RL}\mathrm{I}_{t}^{\varphi}\Phi(t)=\frac{1}{\Gamma(\varphi)}\int_{0}^{t}(t-\rho)^{\varphi-1}\Phi(\rho)d\rho. \tag{2}$$

Definition 2.3. *The constant-proportional Caputo (CPC) fractional operator, a hybrid fractional derivative identified by Dumitru Baleanu et al. [32], is defined by*

$$ {}_{0}^{CPC}\mathrm{D}_t^{\varphi}\Phi(t) = \frac{1}{\Gamma(1-\varphi)} \int_0^t \left[\mathrm{A}_1(\varphi)\Phi(\rho) + \mathrm{A}_0(\varphi)\Phi'(\rho)\right](t-\rho)^{-\varphi} d\rho \tag{3} $$

$$ = \mathrm{A}_1(\varphi)\, {}_{0}^{RL}I_t^{1-\varphi}\Phi(t) + \mathrm{A}_0(\varphi)\, {}_{0}^{C}\mathrm{D}_t^{\varphi}\Phi(t). \tag{4} $$

The CPC integral operator is given as

$$ {}_{0}^{CPC}\mathrm{I}_t^{\varphi}\Phi(t) = \frac{1}{\mathrm{A}_0(\varphi)} \int_0^t \exp\Big\{-\frac{\mathrm{A}_1(\varphi)}{\mathrm{A}_0(\varphi)}(t-\rho)\Big\} {}_{0}^{RL}\mathrm{D}_{\rho}^{1-\varphi}\Phi(\rho) d\rho \tag{5} $$

Theorem 2.1. *[32] The Laplace transform for CPC derivative is given by*

$$ \mathscr{L}\left[{}_{0}^{CPC}\mathrm{D}_t^{\varphi}\Phi(t)\right] = \Big\{\frac{\mathrm{A}_1(\varphi)}{s} + \mathrm{A}_0(\varphi)\Big\} s^{\varphi}\widehat{\Phi}(s) - \mathrm{A}_0 s^{\varphi-1}\Phi(0). \tag{6} $$

Definition 2.1. *Ali Akgül [33] introduced two hybrid fractional operators which are given as follows:*

$$ {}_{0}^{CPABC}\mathrm{D}_t^{\varphi}\Phi(t) = \frac{\mathrm{B}(\varphi)\mathrm{A}_1(\varphi)\Phi(t)}{1-\varphi}\mathscr{E}_{\varphi}\big(-\frac{\varphi}{1-\varphi}t^{\varphi}\big) + \frac{\mathrm{B}(\varphi)\mathrm{A}_0(\varphi)\Phi'(t)}{1-\varphi}\mathscr{E}_{\varphi}\big(-\frac{\varphi}{1-\varphi}t^{\varphi}\big). \tag{7} $$

$$ {}_{0}^{CPCF}\mathrm{D}_t^{\varphi}\Phi(t) = \frac{\mathbb{Q}(\varphi)\mathrm{A}_1(\varphi)\Phi(t)}{1-\varphi}\exp\big(-\frac{\varphi}{1-\varphi}t\big) + \frac{\mathbb{Q}(\varphi)\mathrm{A}_0(\varphi)\Phi'(t)}{1-\varphi}\exp\big(-\frac{\varphi}{1-\varphi}t\big). \tag{8} $$

3. Model Formulation

In addition to presenting a mathematical framework for efficiently distributing mitigation alternatives to reduce energy consumption-related CO_2 emissions, this work attempts to provide a theoretical model that identifies the global relationship between population increase, usage of energy, and carbon dioxide emissions. Here, we go over the three distinct categories that make up the mathematical model [1]:

- Atmospheric CO_2 level (**C**);
- Human population (**N**); and
- Consumed Energy (**E**).

The proposed mathematical model consists of three nonlinear differential equations that relate the amount of CO2 in the atmosphere to the human population and energy usage.

$$ \begin{aligned} \frac{d\mathbf{C}}{dt} &= \eta_1\mathbf{N} + \eta_2(1-\upsilon_2)\mathbf{E} - \delta(\mathbf{C}-\mathrm{C}_0), \\ \frac{d\mathbf{N}}{dt} &= r\mathbf{N}(1-\frac{\mathbf{N}}{\mathrm{L}}) + \mathbf{NE}(\beta_1+\beta_2\mathbf{N}) - \gamma(\mathbf{C}-\mathrm{C}_0)\mathbf{N}, \\ \frac{d\mathbf{E}}{dt} &= (1-\upsilon_1)\frac{\alpha\mathbf{NE}}{K+\mathbf{N}} - \alpha_0\mathbf{E}^2. \end{aligned} \tag{9} $$

Table 1: Interpretation of model parameters

Symbol	Description
η_1	Coefficient of CO_2 emissions from non-energy sector
η_2	Coefficient of CO_2 emissions from energy sector
υ_1	Effectiveness of mitigation strategies to lower the rate of energy use
υ_2	Effectiveness of mitigation strategies to reduce the CO_2 emissions rate per energy unit.
δ	CO_2 sinks removal rate from the atmosphere
r	Internal growth rate
L	Human population's carrying capacity
β_1	Human population's growth rate coefficients
β_2	Human population's carrying capacity as a result of energy consumption
γ	Human population death rate as a result of excessive CO_2 levels
C_0	Initial concentration of CO_2
α	Energy use growth rate
K	Half-saturation constant
α_0	Energy use depletion rate

Fractional calculus is an essential tool in many domains because it enables the realistic simulation of genuine occurrences that depend on both the past and present time history. Under the constant-proportional Caputo (CPC) type fractional derivative with $0 < \varsigma \leq 1$, we modify the system (9) by using the nonlinear fractional differential equations listed below.

$$\begin{cases} {}^{CPC}{}_0D_t^{\varsigma}\mathbf{C}(t) &= \eta_1\mathbf{N} + \eta_2(1-\upsilon_2)\mathbf{E} - \delta(\mathbf{C} - \mathrm{C}_0), \\ {}^{CPC}{}_0D_t^{\varsigma}\mathbf{N}(t) &= r\mathbf{N}(1-\frac{\mathbf{N}}{\mathrm{L}}) + \mathbf{NE}(\beta_1 + \beta_2\mathbf{N}) - \gamma(\mathbf{C} - \mathrm{C}_0)\mathbf{N}, \\ {}^{CPC}{}_0D_t^{\varsigma}\mathbf{E}(t) &= (1-\upsilon_1)\frac{\alpha\mathbf{NE}}{K+\mathbf{N}} - \alpha_0\mathbf{E}^2, \end{cases} \tag{10}$$

where the initial conditions are given by

$$\mathrm{C}(0),\ \mathrm{N}(0),\ \mathrm{E}(0) \geq 0. \tag{11}$$

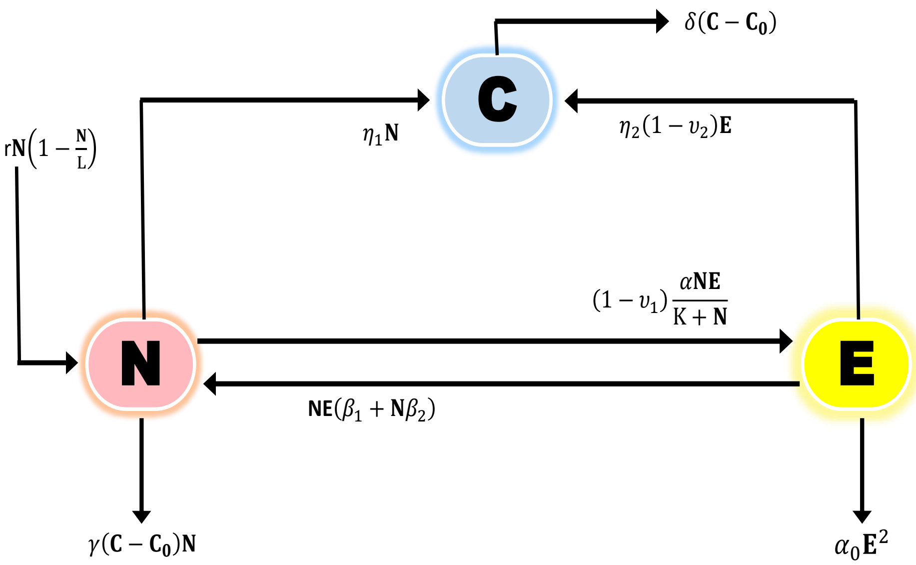

Figure 1: Dynamical model

4. Qualitative Analysis of Proposed Model

4.1. Well-posedness

Theorem 4.1. *The solution of the proposed model (10) is distinct and limited to $\mathbb{R}^3_+$ in given initial conditions.*

Proof. We will show the affirmative solution of the system (10), and the results are as follows:

$$\begin{aligned} {}^{CPC}{}_0D_t^{\varsigma}\mathbf{C}(t)\big|_{\mathbf{C}=0} &= \eta_1\mathbf{N}+\eta_2(1-\upsilon_2)\mathbf{E}\geq 0, \\ {}^{CPC}{}_0D_t^{\varsigma}\mathbf{N}(t)\big|_{\mathbf{N}=0} &= 0, \\ {}^{CPC}{}_0D_t^{\varsigma}\mathbf{E}(t)\big|_{\mathbf{E}=0} &= 0. \end{aligned} \tag{12}$$

The domain is positivity invariant since the solution is unable to evacuate the hyperplane if

$$(\mathbf{C}(0),\mathbf{N}(0),\mathbf{E}(0))\in\mathbb{R}^3_+.$$

The feasible region is given by

$$\varpi=\Big\{(\mathbf{N},\ \mathbf{C},\ \mathbf{E})\in\mathbb{R}^3_+ : \mathbf{C}\in[\mathbf{C_0},\mathbf{C_q}],\ \mathbf{N}\in[0,\mathbf{N_q}],\ \mathbf{E}\in[0,\mathbf{E_q}]\Big\}, \tag{13}$$

where

$$\mathbf{C_q}=\mathrm{C}_0+\frac{\eta_1\mathbf{N_q}+\eta_2(1-\upsilon_2)\mathbf{E}_q}{\delta},$$

$$\mathbf{N_q}=\frac{\mathrm{L}(r+\beta_1\mathbf{E_q})}{r-\mathrm{L}\beta_2\mathbf{E_q}},$$

and

$$\mathbf{E_q}=\frac{\alpha(1-\upsilon_1)}{\alpha_0}.$$

□

4.2. Equilibrium points

A steady solution of a dynamical system that doesn't change over time is called an equilibrium state (P). We need to set the equations of the system (10) to zero in order to find equilibrium locations. A list of model (10)'s viable equilibria is provided.

$$P_1=\{\mathrm{C}_0,\ 0,\ 0\},$$

$$P_2=\{\mathrm{C}_0+\frac{\eta_1 r\delta\mathrm{L}}{r\delta+\eta_1\gamma\mathrm{L}},\ \frac{r\delta\mathrm{L}}{r\delta+\eta_1\gamma\mathrm{L}},\ 0\},$$

and

$$P_3=\{\mathbf{N}^*,\ \mathbf{C}^*,\ \mathbf{E}^*\}.$$

It is evident that P_1 and P_2 exist. The elements $\mathbf{N}^*$, $\mathbf{C}^*$, and $\mathbf{E}^*$ in equilibrium P_3 are positive solutions to the subsequent equations:

$$\begin{aligned} \eta_1\mathbf{N}+\eta_2(1-\upsilon_2)\mathbf{E}-\delta(\mathbf{C}-\mathrm{C}_0) &= 0, \\ r(1-\frac{\mathbf{N}}{\mathrm{L}})+\mathbf{E}(\beta_1+\beta_2\mathbf{N})-\gamma(\mathbf{C}-\mathrm{C}_0) &= 0, \\ (1-\upsilon_1)\frac{\alpha\mathbf{N}}{K+\mathbf{N}}-\alpha_0\mathbf{E} &= 0. \end{aligned} \tag{14}$$

5. CPC integral operator

Proposition 3.1: The CPC integral is defined [32] as:

$$\begin{cases} {}^{CPC}\mathrm{I}_t^{\varsigma}\mathbf{N}(t) = \frac{1}{\mathrm{A}_0(\varsigma)}\int_0^t \exp\left[-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}(t-\rho)\right]{}^{RL}\mathbb{D}_\rho^{1-\varsigma}\mathbf{N}(\rho)d\rho, \\ {}^{CPC}\mathrm{I}_t^{\varsigma}\mathbf{C}(t) = \frac{1}{\mathrm{A}_2(\varsigma)}\int_0^t \exp\left[-\frac{\mathrm{A}_3(\varsigma)}{\mathrm{A}_2(\varsigma)}(t-\rho)\right]{}^{RL}\mathbb{D}_\rho^{1-\varsigma}\mathbf{C}(\rho)d\rho, \\ {}^{CPC}\mathrm{I}_t^{\varsigma}\mathbf{E}(t) = \frac{1}{\mathrm{A}_4(\varsigma)}\int_0^t \exp\left[-\frac{\mathrm{A}_5(\varsigma)}{\mathrm{A}_4(\varsigma)}(t-\rho)\right]{}^{RL}\mathbb{D}_\rho^{1-\varsigma}\mathbf{E}(\rho)d\rho. \end{cases} \tag{15}$$

This ensures:

$$\begin{cases} {}^{CPC}\mathbb{D}_t^{\varsigma}\,{}^{PC}\mathrm{I}_t^{\varsigma}\mathbf{N}(t) = \mathbf{N}(t) - \frac{t^{-\varsigma}}{\Gamma(1-\varsigma)}\lim_{t\to 0}{}^{RL}\mathrm{I}_t^{\varsigma}\mathbf{N}(t), \\ {}^{CPC}\mathbb{D}_t^{\varsigma}\,{}^{PC}\mathrm{I}_t^{\varsigma}\mathbf{C}(t) = \mathbf{C}(t) - \frac{t^{-\varsigma}}{\Gamma(1-\varsigma)}\lim_{t\to 0}{}^{RL}\mathrm{I}_t^{\varsigma}\mathbf{C}(t), \\ {}^{CPC}\mathbb{D}_t^{\varsigma}\,{}^{PC}\mathrm{I}_t^{\varsigma}\mathbf{E}(t) = \mathbf{E}(t) - \frac{t^{-\varsigma}}{\Gamma(1-\varsigma)}\lim_{t\to 0}{}^{RL}\mathrm{I}_t^{\varsigma}\mathbf{E}(t), \end{cases} \tag{16}$$

and

$$\begin{cases} {}^{CPC}\mathrm{I}_t^{\varsigma}\,{}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{N}(t) = \mathbf{N}(t) - \exp\left[-\int_0^t \frac{\mathrm{A}_1(\varsigma,\mu)}{\mathrm{A}_0(\varsigma,\mu)}d\mu\right]\mathbf{N}(0), \\ {}^{CPC}\mathrm{I}_t^{\varsigma}\,{}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{C}(t) = \mathbf{C}(t) - \exp\left[-\int_0^t \frac{\mathrm{A}_3(\varsigma,\mu)}{\mathrm{A}_2(\varsigma,\mu)}d\mu\right]\mathbf{C}(0), \\ {}^{CPC}\mathrm{I}_t^{\varsigma}\,{}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{E}(t) = \mathbf{E}(t) - \exp\left[-\int_0^t \frac{\mathrm{A}_5(\varsigma,\mu)}{\mathrm{A}_4(\varsigma,\mu)}d\mu\right]\mathbf{E}(0). \end{cases} \tag{17}$$

Proof. We can prove that

$$\begin{cases} \left({}^{PC}\mathbb{D}_t^{\varsigma}\bullet{}^{PC}\mathrm{I}_t^{\varsigma}\right)\mathbf{N}(t) &= \left({}^{RL}\mathrm{I}_t^{1-\varsigma}\bullet{}^{P}\mathbb{D}_t^{\varsigma}\right)\bullet\left({}^{P}\mathrm{I}_t^{\varsigma}\bullet{}^{RL}\mathbb{D}_t^{1-\varsigma}\right)\mathbf{N}(t) = \left({}^{RL}\mathrm{I}_t^{1-\varsigma}\bullet{}^{RL}\mathbb{D}_t^{1-\varsigma}\right)\mathbf{N}(t) \\ &= \mathbf{N}(t) - \frac{t^{-\varsigma}}{\Gamma(1-\varsigma)}\lim_{t\to 0}{}^{RL}\mathrm{I}_t^{\varsigma}\mathbf{N}(t) = \left({}^{P}\mathrm{I}_t^{\varsigma}\bullet{}^{RL}\mathbb{D}_t^{1-\varsigma}\right)\bullet\left({}^{RL}\mathrm{I}_t^{1-\varsigma}\bullet{}^{P}\mathbb{D}_t^{\varsigma}\right)\mathbf{N}(t) \\ &= \left({}^{P}\mathrm{I}_t^{\varsigma}\bullet{}^{P}\mathbb{D}_t^{\varsigma}\right)\mathbf{N}(t) = \mathbf{N}(t) - \exp\left(-\int_0^t \frac{\mathrm{A}_1(\varsigma,\mu)}{\mathrm{A}_0(\varsigma,\mu)}d\mu\right)\mathbf{N}(0). \end{cases} \tag{18}$$

□

We may invert the CPC operator by combining the following two ideas: the Laplace transform and the results of the aforementioned preposition. Let $\mathbf{N}(0) = \mathbf{C}(0) = \mathbf{E}(0) = 0$, then

$$\begin{cases} \mathscr{L}\left({}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{N}(t)\right) = \left(\frac{\mathrm{A}_1(\varsigma)}{N} + \mathrm{A}_0(\varsigma)\right)S^{\varsigma}\widehat{\mathbf{N}}(S) = \mathrm{A}_0(\varsigma)\left(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}S^{-1}\right)S^{\varsigma}\widehat{\mathbf{N}}(S), \\ \mathscr{L}\left({}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{C}(t)\right) = \left(\frac{\mathrm{A}_3(\varsigma)}{N} + \mathrm{A}_2(\varsigma)\right)S^{\varsigma}\widehat{\mathbf{C}}(S) = \mathrm{A}_2(\varsigma)\left(1+\frac{\mathrm{A}_3(\varsigma)}{\mathrm{A}_2(\varsigma)}S^{-1}\right)S^{\varsigma}\widehat{\mathbf{C}}(S), \\ \mathscr{L}\left({}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{E}(t)\right) = \left(\frac{\mathrm{A}_5(\varsigma)}{N} + \mathrm{A}_4(\varsigma)\right)S^{\varsigma}\widehat{\mathbf{E}}(S) = \mathrm{A}_4(\varsigma)\left(1+\frac{\mathrm{A}_5(\varsigma)}{\mathrm{A}_4(\varsigma)}S^{-1}\right)S^{\varsigma}\widehat{\mathbf{E}}(S). \end{cases} \tag{19}$$

Therefore, writing ${}^{CPC}\mathbb{D}_t^{\varsigma}\mathbf{N}(t) = g_1(t)$, we have

$$\begin{aligned} \widehat{\mathbf{N}}(S) &= \left\{\mathrm{A}_0(\varsigma)\left(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}S^{-1}\right)S^{\varsigma}\right\}^{-1}\widehat{g_1}(S) \\ &= \frac{1}{\mathrm{A}_0(\varsigma)}S^{-\varsigma}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}S^{-1}\right)^n\widehat{g_1}(S) = \sum_{n=0}^{\infty}\frac{\{-\mathrm{A}_1(\varsigma)\}^n}{\mathrm{A}_0(\varsigma)^{n+1}}S^{-\varsigma-n}\widehat{g_1}(S). \end{aligned} \tag{20}$$

Only the conditions $\left|\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}S^{-\varsigma}\right| < 1$. We have the series formula as follows from equation (20):

$$\begin{cases} \mathbf{N}(t) = \sum_{n=0}^{\infty}\frac{(-\mathrm{A}_1(\varsigma))^n}{\mathrm{A}_0(\varsigma)^{n+1}}{}^{RL}\mathrm{I}_t^{\varsigma+n}g_1(t), \\ \mathbf{C}(t) = \sum_{n=0}^{\infty}\frac{(-\mathrm{A}_3(\varsigma))^n}{\mathrm{A}_2(\varsigma)^{n+1}}{}^{RL}\mathrm{I}_t^{\varsigma+n}g_2(t), \\ \mathbf{E}(t) = \sum_{n=0}^{\infty}\frac{(-\mathrm{A}_5(\varsigma))^n}{\mathrm{A}_4(\varsigma)^{n+1}}{}^{RL}\mathrm{I}_t^{\varsigma+n}g_3(t). \end{cases} \tag{21}$$

The 2^{nd} method is

$$\widehat{\mathbf{N}}(S)=\Big(\sum_{n=0}^{\infty}\frac{(-\mathrm{A}_1(\varsigma))^n}{\mathrm{A}_0(\varsigma)^{n+1}}S^{-\varsigma-n}\Big)\widehat{g_1}(S)=\mathscr{L}\Big(\sum_{n=0}^{\infty}\frac{(-\mathrm{A}_1(\varsigma))^n}{\mathrm{A}_0(\varsigma)^{n+1}}.\frac{t^{\varsigma+n-1}}{\Gamma(\varsigma+n)}\Big)\widehat{g_1}(S)$$
$$=\mathscr{L}\Big(\frac{t^{\varsigma-1}}{\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left\{\frac{-\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right\}^n.\frac{1}{\Gamma(\varsigma+n)}\Big)\widehat{g_1}(S)=\mathscr{L}\Big(\frac{t^{\varsigma-1}}{\mathrm{A}_0(\varsigma)}\mathrm{C}_{1,\varsigma}\left\{\frac{-\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}t\right\}\Big)\widehat{g_1}(S), \tag{22}$$

where

$$\mathrm{C}_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\alpha+\beta)}.$$

6. Model analysis with other hybrid fractional derivatives

6.1. CPABC operator

Theorem 6.1. *Let*

$$\begin{cases}{}_0^{CPABC}D_t^{\varsigma}\mathbf{N}(t)=\mathbb{G}_1(t)\\ {}_0^{CPABC}D_t^{\varsigma}\mathbf{C}(t)=\mathbb{G}_2(t)\\ {}_0^{CPABC}D_t^{\varsigma}\mathbf{E}(t)=\mathbb{G}_3(t).\end{cases} \tag{23}$$

Appllying Laplace Transform and letting $\mathbf{N}(0)=\mathbf{C}(0)=\mathbf{E}(0)=0$ *results in*

$$\begin{cases}\mathbf{N}(t)=\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{\varsigma+n}\mathbb{G}_1(t)+\frac{1-\varsigma}{\mathsf{M}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{n}\mathbb{G}_1(t),\\ \mathbf{C}(t)=\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{\varsigma+n}\mathbb{G}_2(t)+\frac{1-\varsigma}{\mathsf{M}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{n}\mathbb{G}_2(t),\\ \mathbf{E}(t)=\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{\varsigma+n}\mathbb{G}_3(t)+\frac{1-\varsigma}{\mathsf{M}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\Big)^n{}_0I_t^{n}\mathbb{G}_3(t).\end{cases} \tag{24}$$

Proof. We have

$$\begin{cases}\mathscr{L}\big[\mathbf{N}(t)\big]\left(\frac{\mathrm{B}(\varsigma)\mathrm{A}_1(\varsigma)s^{\varsigma-1}+s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s^{\varsigma}(1-\varsigma)}\right)=\mathscr{L}\big[\mathbb{G}_1(t)\big],\\ \mathscr{L}\big[\mathbf{C}(t)\big]\left(\frac{\mathrm{B}(\varsigma)\mathrm{A}_1(\varsigma)s^{\varsigma-1}+s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s^{\varsigma}(1-\varsigma)}\right)=\mathscr{L}\big[\mathbb{G}_2(t)\big],\\ \mathscr{L}\big[\mathbf{E}(t)\big]\left(\frac{\mathrm{B}(\varsigma)\mathrm{A}_1(\varsigma)s^{\varsigma-1}+s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s^{\varsigma}(1-\varsigma)}\right)=\mathscr{L}\big[\mathbb{G}_3(t)\big].\end{cases} \tag{25}$$

We can express above relation as:

$$\begin{cases}\mathscr{L}\big[\mathbf{N}(t)\big]\left(\frac{\mathrm{B}(\varsigma)s^{\varsigma-1}}{\varsigma+s^{\varsigma}(1-\varsigma)}\Big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\Big)\right)=\mathscr{L}\big[\mathbb{G}_1(t)\big],\\ \mathscr{L}\big[\mathbf{C}(t)\big]\left(\frac{\mathrm{B}(\varsigma)s^{\varsigma-1}}{\varsigma+s^{\varsigma}(1-\varsigma)}\Big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\Big)\right)=\mathscr{L}\big[\mathbb{G}_2(t)\big],\\ \mathscr{L}\big[\mathbf{E}(t)\big]\left(\frac{\mathrm{B}(\varsigma)s^{\varsigma-1}}{\varsigma+s^{\varsigma}(1-\varsigma)}\Big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\Big)\right)=\mathscr{L}\big[\mathbb{G}_3(t)\big],\end{cases} \tag{26}$$

which equals

$$\begin{cases}\mathscr{L}\big[\mathbf{N}(t)\big]=\frac{\varsigma+s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma-1}\big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\big)}\mathscr{L}\big[\mathbb{G}_1(t)\big],\\ \mathscr{L}\big[\mathbf{C}(t)\big]=\frac{\varsigma+s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma-1}\big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\big)}\mathscr{L}\big[\mathbb{G}_2(t)\big],\\ \mathscr{L}\big[\mathbf{E}(t)\big]=\frac{\varsigma+s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma-1}\big(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma)\big)}\mathscr{L}\big[\mathbb{G}_3(t)\big].\end{cases} \tag{27}$$

$$\begin{cases} = \frac{\varsigma+s^{\varsigma}(1-\varsigma)}{s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)\left(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)}\mathscr{L}\left[\mathbb{G}_1(t)\right], \\ = \frac{\varsigma+s^{\varsigma}(1-\varsigma)}{s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)\left(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)}\mathscr{L}\left[\mathbb{G}_2(t)\right], \\ = \frac{\varsigma+s^{\varsigma}(1-\varsigma)}{s^{\varsigma}\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)\left(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)}\mathscr{L}\left[\mathbb{G}_3(t)\right]. \end{cases} \tag{28}$$

$$\begin{cases} = \left(\frac{\varsigma}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n + \frac{s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n\right)\mathscr{L}[\mathbb{G}_1(t)], \\ = \left(\frac{\varsigma}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n + \frac{s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n\right)\mathscr{L}[\mathbb{G}_2(t)], \\ = \left(\frac{\varsigma}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n + \frac{s^{\varsigma}(1-\varsigma)}{\mathrm{B}(\varsigma)s^{\varsigma}\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\right)^n\right)\mathscr{L}[\mathbb{G}_3(t)]. \end{cases} \tag{29}$$

$$\begin{cases} = \left[\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n-\varsigma} + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n}\right]\mathscr{L}[\mathbb{G}_1(t)], \\ = \left[\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n-\varsigma} + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n}\right]\mathscr{L}[\mathbb{G}_2(t)], \\ = \left[\frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n-\varsigma} + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n s^{-n}\right]\mathscr{L}[\mathbb{G}_3(t)]. \end{cases} \tag{30}$$

$$\begin{cases} = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{\varsigma+n}\mathbb{G}_1(t)] + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{n}\mathbb{G}_1(t)], \\ = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{\varsigma+n}\mathbb{G}_2(t)] + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{n}\mathbb{G}_2(t)], \\ = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{\varsigma+n}\mathbb{G}_3(t)] + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n \mathscr{L}[{}_0I_t^{n}\mathbb{G}_3(t)]. \end{cases} \tag{31}$$

Applying Inverse Laplace transform, we achieve

$$\begin{cases} \mathbf{N}(t) = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{\varsigma+n}\mathbb{G}_1(t) + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{n}\mathbb{G}_1(t), \\ \mathbf{C}(t) = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{\varsigma+n}\mathbb{G}_2(t) + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{n}\mathbb{G}_2(t), \\ \mathbf{E}(t) = \frac{\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{\varsigma+n}\mathbb{G}_3(t) + \frac{1-\varsigma}{\mathrm{B}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\right)^n {}_0I_t^{n}\mathbb{G}_3(t). \end{cases} \tag{32}$$

□

6.2. CPCF operator

Theorem 6.1. *[33] Consider the following system of differential equations with the CPCF operator as:*

$$\begin{cases} {}_0^{CPCF}\mathrm{D}_t^{\varsigma}\mathbf{N}(t) = \mathrm{Y}_1(t), \\ {}_0^{CPCF}\mathrm{D}_t^{\varsigma}\mathbf{C}(t) = \mathrm{Y}_2(t), \\ {}_0^{CPCF}\mathrm{D}_t^{\varsigma}\mathbf{E}(t) = \mathrm{Y}_3(t). \end{cases} \tag{33}$$

Utilizing Laplace Transform on system (33) and assuming $\mathbf{N}(0)=\mathbf{C}(0)=\mathbf{E}(0)=0$ *yields in*

$$\begin{cases}\mathbf{N}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_1(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_1(t),\\ \mathbf{C}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_2(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_2(t),\\ \mathbf{E}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_3(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_3(t).\end{cases}\tag{34}$$

Proof. We have

$$\begin{cases}\mathscr{L}(\mathbf{N}(t))\big(\frac{\mathbb{Q}(\varsigma)\mathrm{A}_1(\varsigma)}{\varsigma+s(1-\varsigma)}+\frac{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s(1-\varsigma)}\big)=\mathscr{L}(\mathbf{N}(t))\big(\frac{\mathbb{Q}(\varsigma)}{\varsigma+s(1-\varsigma)}(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))\big)=\mathscr{L}(\mathrm{Y}_1(t)),\\ \mathscr{L}(\mathbf{C}(t))\big(\frac{\mathbb{Q}(\varsigma)\mathrm{A}_1(\varsigma)}{\varsigma+s(1-\varsigma)}+\frac{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s(1-\varsigma)}\big)=\mathscr{L}(\mathbf{C}(t))\big(\frac{\mathbb{Q}(\varsigma)}{\varsigma+s(1-\varsigma)}(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))\big)=\mathscr{L}(\mathrm{Y}_2(t)),\\ \mathscr{L}(\mathbf{E}(t))\big(\frac{\mathbb{Q}(\varsigma)\mathrm{A}_1(\varsigma)}{\varsigma+s(1-\varsigma)}+\frac{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}{\varsigma+s(1-\varsigma)}\big)=\mathscr{L}(\mathbf{E}(t))\big(\frac{\mathbb{Q}(\varsigma)}{\varsigma+s(1-\varsigma)}(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))\big)=\mathscr{L}(\mathrm{Y}_3(t)).\end{cases}\tag{35}$$

$$\begin{cases}\mathscr{L}(\mathbf{N}(t))=\frac{\varsigma+s(1-\varsigma)}{\mathbb{Q}(\varsigma)(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))}\mathscr{L}(\mathrm{Y}_1(t))=\frac{\varsigma+s(1-\varsigma)}{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1})}\mathscr{L}(\mathrm{Y}_1(t)),\\ \mathscr{L}(\mathbf{C}(t))=\frac{\varsigma+s(1-\varsigma)}{\mathbb{Q}(\varsigma)(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))}\mathscr{L}(\mathrm{Y}_2(t))=\frac{\varsigma+s(1-\varsigma)}{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1})}\mathscr{L}(\mathrm{Y}_2(t)),\\ \mathscr{L}(\mathbf{E}(t))=\frac{\varsigma+s(1-\varsigma)}{\mathbb{Q}(\varsigma)(\mathrm{A}_1(\varsigma)+s\mathrm{A}_0(\varsigma))}\mathscr{L}(\mathrm{Y}_3(t))=\frac{\varsigma+s(1-\varsigma)}{s\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)(1+\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1})}\mathscr{L}(\mathrm{Y}_3(t)).\end{cases}\tag{36}$$

$$\begin{cases}\mathscr{L}(\mathbf{N}(t))=\frac{\varsigma s^{-1}}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_1(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_1(t)),\\ \mathscr{L}(\mathbf{C}(t))=\frac{\varsigma s^{-1}}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_2(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_2(t)),\\ \mathscr{L}(\mathbf{E}(t))=\frac{\varsigma s^{-1}}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_3(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}s^{-1}\big)^n\mathscr{L}(\mathrm{Y}_3(t)).\end{cases}\tag{37}$$

$$\begin{cases}=\Big(\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n-1}+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n}\Big)\mathscr{L}(\mathrm{Y}_1(t)),\\ =\Big(\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n-1}+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n}\Big)\mathscr{L}(\mathrm{Y}_2(t)),\\ =\Big(\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n-1}+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n s^{-n}\Big)\mathscr{L}(\mathrm{Y}_3(t)).\end{cases}\tag{38}$$

$$\begin{cases}=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n+1}\mathrm{Y}_1(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n}\mathrm{Y}_1(t)),\\ =\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n+1}\mathrm{Y}_2(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n}\mathrm{Y}_2(t)),\\ =\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n+1}\mathrm{Y}_3(t))+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n\mathscr{L}({}_0\mathrm{I}_t^{n}\mathrm{Y}_3(t)).\end{cases}\tag{39}$$

Inverse Laplace transform results finally in

$$\begin{cases}\mathbf{N}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_1(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_1(t),\\ \mathbf{C}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_1(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_2(t),\\ \mathbf{E}(t)=\frac{\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n+1}\mathrm{Y}_1(t)+\frac{1-\varsigma}{\mathbb{Q}(\varsigma)\mathrm{A}_0(\varsigma)}\sum_{n=0}^{\infty}\big(-\frac{\mathrm{A}_1(\varsigma)}{\mathrm{A}_0(\varsigma)}\big)^n{}_0\mathrm{I}_t^{n}\mathrm{Y}_3(t).\end{cases}\tag{40}$$

□

7. Conclusion

The demand for energy is rising due to population growth and the burning of fossil fuels, which raises greenhouse gas emissions. Reducing emissions from the energy industry is essential to addressing climate change. This study examines carbon emissions from the power industry using hybrid fractional operators. These operators alter differential equations to produce a mathematical model that characterizes the amount of carbon dioxide in the atmosphere, human population, and energy consumption. Compartmental models are crucial for comprehending real-world occurrences because they offer a mathematical framework for comprehending systems and predicting results. The dynamical process is strongly affected by the order of the fractional derivative.

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